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Viscous effects on fully coupled resonant-triad interactions: an analytical approach

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This paper is concerned with viscous effects on the development of a fully coupled resonant triad consisting of Rayleigh waves. Complementary to the numerical study of Lee (1995), we attack this problem analytically. The fully coupled amplitude equations are derived with all the kernels involved being expressed in closed forms. The amplitude equations are then solved numerically. It is found that viscosity reduces the growth of the disturbance in the parametric-resonance stage and delays the final occurrence of the finite-time singularity. But viscosity does not appear to be able to eliminate the singularity. While the analysis is performed for the temporally evolving instability waves, we demonstrate its broad application by showing that it can be slightly modified to obtain the amplitude equations for the spatially growing Rayleigh waves, and the equations which describe the development of the resonant-triad of Tollmien–Schlichting waves in the fully interactive stage.

1. Introduction

Resonant-triad interaction is an important mechanism which can operate in a variety of wave-flow motions. Of special interest is the so-called subharmonic resonant triad which consists of a fundamental two-dimensional wave and a pair of three-dimensional waves with subharmonic frequency. In particular, the subharmonic resonant-triad interaction of the Tollmien–Schlichting waves has been proposed by Raetz (1959) and Craik (1971) to be one of the mechanisms that induce rapid growth of the three-dimensional disturbances in laminar–turbulence transitions in boundary layers. Following the experimental observations of Kachanov & Levchenko (1984) and Saric & Thomas (1984), the role of subharmonic resonance in causing boundary layer transition has been generally accepted. For an introduction to background and earlier studies, the reader is referred to Craik (1985).

The paper of Smith & Stewart (1987) represents the first attempt to put earlier studies on a completely asymptotic base by using a systematic high-Reynolds-number approach. They show that for *Tollmien–Schlichting* (TS) waves, the subharmonic resonance consisting of three nearly neutral modes can occur at some distance downstream of lower branch of the neutral curve, i.e. in the so-called high-frequency limit of the lower-branch-scaling regime. The resonance condition is that the two subharmonic waves must travel at angles of $\pm 60^{\circ}$ to the streamwise direction, as found earlier by Craik (1971) for the resonant triad of Rayleigh waves. Since resonant interactions are observed to take place near, or downstream of, the upper-branch of the neutral curve, Mankbadi, Wu & Lee (1993) and Wu (1993) studied the resonant triad in the upper-branch-scaling regime for the Blasius and for the accelerating boundary layers respectively. They show that dominant nonlinear effects come from the critical layers and the surrounding diffusion layers. This in turn casts some doubt on Smith & Stewart's (1987) assumption that the critical layer is passive in their case (see also Khokhlov 1993). Jennings, Stewart & Wu (1995) are currently investigating this issue in detail. They find that the critical layer and the diffusion layers indeed play an active role. By taking the nonlinearity associated with the critical layer and diffusion layer into account, they show that the interaction can occur at a lower amplitude than that in Smith & Stewart (1987).

The evolution of a resonant triad consisting of slowly modulated *Rayleigh* instability modes has been studied by Goldstein & Lee (1992) and Wu (1992) for the long-wavelength disturbance and for the order-one wavelength (nearly neutral) disturbance respectively. As is appropriate for Rayleigh waves, they fix their asymptotic scalings so that the critical layers involved are of non-equilibrium type, in contrast to the equilibrium, viscous-dominated critical layers in Mankbadi *et al.* (1993) and Wu (1993). In addition, by choosing suitable amplitudes for the disturbances, the development of the oblique modes is affected by the quadratic interaction, and at the same time the oblique modes produce a back effect on the planar mode. Because of this feature, Goldstein & Lee (1992) refer to this type of triad as 'fully coupled' or 'fully interactive'. It is shown that nonlinear effects can cause a singularity within a finite time (or distance). However, this conclusion is obtained after viscosity is completely ignored.

In the related context, the effects of viscosity on the development of disturbances are found to be rather subtle. For instance, a two-dimensional disturbance with a regular critical layer saturates in an oscillatory manner if viscosity is neglected (Goldstein & Leib 1988). But when viscosity is included, the disturbance grows algebraically no matter how small the viscosity is (Goldstein & Hultgren 1989). For a pair of oblique modes, Wu, Lee & Cowley (1993) show that while the disturbance develops a finite-time singularity in the purely inviscid case (Goldstein & Choi 1989), it can decay exponentially when viscosity is sufficiently large.

The viscous effect on the development of the fully coupled resonant triad was first investigated by Lee (1995), who solved the appropriate partial differential equations governing the flow in the critical layer numerically. In this paper, however, we shall attack this problem analytically. As well as being complementary to the numerical study of Lee (1995), our present undertaking is also desirable in view of the wide application of the results. The final amplitude equations apply to a broad class of shear flows which can support Rayleigh instability waves (e.g. free shear layers); the kernels are calculated once and for all for these flows. Moreover, Goldstein (1994) recently has observed that for the resonant triad of TS waves, the parametric resonance can lead to a stage at which the critical layer can become both of viscous and non-equilibrium type. He also shows that by choosing the appropriate initial magnitude for oblique waves, the interaction can become fully coupled. This is in contrast to the viscous-dominated equilibrium critical layer regime of Mankbadi *et al.* (1993) where the interaction is not fully coupled in that the subharmonics have no back effect on the planar mode. Goldstein (1994) showed that the analytical results obtained in this paper would be applicable to the development of the resonant triad of the TS waves (see §5.2 below).

The paper is organized as follows. In the next section, we formulate the problem using the Stokes oscillatory layer as an example. The solution in the main part of the flow is then considered. Because this part of the analysis, to the order of interest of this study, is exactly the same as in the inviscid case (Wu 1992), we shall only outline the main results. In §3, the flow within the viscous, non-equilibrium critical layers is analysed. The solutions are found analytically. Matching them onto those in the outer region, we obtain the coupled amplitude equations (§4). The kernels involved are expressed in closed forms. In §5, by a minor modification of our analysis, we derive the amplitude equations for the resonant triad of Rayleigh waves in spatially developing shear layers as well as for the resonant triad of Tollmien–Schlichting waves in the Blasius boundary layer. In §6, we solve the amplitude equations numerically, and discuss the results.

2. Formulation and outer expansions

As in Wu (1992), the flow is described in terms of Cartesian coordinates $(x^*, y^*, z^*) = \delta^*(x, y, z)$, where x^* is parallel to the direction of oscillation of the plate, y^* is normal to the plate and z^* is the spanwise direction. We non-dimensionalize time with ω^{-1}, i.e. $\tau = \omega t^*$, and write the velocity as $U_0(U, V, W)$, where ω and U_0 are the dimensional oscillation frequency and the maximum velocity of the plate, and $\delta^* = (2\nu/\omega)^{1/2}$ represents the thickness of the Stokes layer. The pressure is non-dimensionalized by $\rho_0 U_0^2$, where ρ_0 is the density of the fluid. The Reynolds number based on U_0 and δ^* is $R = U_0(2/\nu\omega)^{1/2}$, where ν is the kinematic viscosity. The analysis actually applies to any almost parallel, inviscidly unstable flow $(\bar{U}, R^{-1}\bar{V}, 0)$. However, for purposes of illustration we will substitute at appropriate points the Stokes-layer solution for the flow over an oscillating plate:

$$(\bar{U}, R^{-1}\bar{V}, 0) = (\cos(\tau - y)e^{-y}, 0, 0) .$$

We denote the velocity of the perturbed flow by

$$(U, V, W) = (\bar{U} + u, R^{-1}\bar{V} + v, w) .$$

The disturbance consists of a fundamental planar mode with a magnitude δ and a pair of subharmonic oblique modes with a magnitude $\epsilon = O(\delta^{3/4})$ (see below). For simplicity, we shall assume that the two oblique modes are of equal amplitude. In principle, it is straightforward to extend the analysis to unequal amplitudes. However, the asymmetry in the amplitudes would complicate the algebra considerably.

As usual in weakly nonlinear theory, we assume that nonlinear effects first come into play when the disturbance is nearly neutral, say near a neutral time τ_0. As explained in detail by Wu (1992), it is appropriate to concentrate on times close to

$$\tau = \tau_0 + \epsilon^{1/3}\tau_1 ,$$

for some suitable $\tau_1 = O(1)$, i.e. times at which the linear growth rate is $O(\epsilon^{1/3}R)$. As in Wu (1992), we introduce the time scales

$$t_1 = \tfrac{1}{2}\epsilon^{1/3}R\tau , \tag{2.1}$$

and

$$t = R\tau \tag{2.2}$$

to account for the nonlinear development and the carrier wave frequency of the disturbance, respectively. The basic-flow velocity $\bar{U}$ evolves on the very slow time scale τ, and it is sufficient to express its profile at time τ as a Taylor series about the neutral time τ_0:

$$\bar{U}(y, \tau) = \bar{U}(y, \tau_0) + \epsilon^{1/3}\bar{U}_\tau(y, \tau_0)\tau_1 + \dots .$$

In order to investigate the viscous effects, we consider the distinguished case where viscous diffusion terms appear at leading order in the critical-layer equations. This occurs when

$$R^{-1} = \lambda\epsilon \,, \tag{2.3}$$

where the parameter λ characterizes the relative importance of viscous to unsteady inertial effects, and thus to nonlinear effects (cf. Haberman 1972). Throughout §§2–4, λ will be assumed to be of order one.

Outside the critical layers, the unsteady flow is basically linear and inviscid. The expansion for the velocity (u, v, w) and the pressure p of the disturbance is the same as in Wu (1992), namely

$$u = \epsilon u_1 + \epsilon^{4/3} u_2 + \epsilon^{5/3} u_3 + \dots \,, \tag{2.4}$$

$$v = \epsilon v_1 + \epsilon^{4/3} v_2 + \epsilon^{5/3} v_3 + \dots \,, \tag{2.5}$$

$$w = \epsilon w_1 + \epsilon^{4/3} w_2 + \epsilon^{5/3} w_3 + \dots \,, \tag{2.6}$$

$$p = \epsilon p_1 + \epsilon^{4/3} p_2 + \epsilon^{5/3} p_3 + \dots \,. \tag{2.7}$$

Guided by the 'early time' linear solution, we seek the solution of the form

$$v_1 = A(t_1)\bar{v}_1(y)\cos\beta z E + \text{c.c.} \,, \tag{2.8}$$

where $A(t_1)$ is the amplitude function of the oblique waves, and $\bar{v}_1$ is the eigenfunction of Rayleigh's equation. For convenience, we have defined

$$E = \exp(\mathrm{i}\alpha x - \mathrm{i}\hat{\theta}(t)) \,, \quad \frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t} = \tfrac{1}{2}\alpha c(\tau_0) + \tfrac{1}{2}\lambda^{1/2}\epsilon^{1/2}\Omega_1(\tau_0) + \dots \,, \tag{2.9}$$

where α and β are the imposed streamwise and spanwise wavenumbers; in the case of a steady non-parallel shear layer, the form of the solution has to be changed slightly so that the frequency and spanwise wavenumber are imposed. The condition for the disturbance to form a subharmonic resonant triad is (see e.g. Craik 1971; Goldstein & Lee 1992; Wu 1992)

$$\beta = \sqrt{3}\alpha \,. \tag{2.10}$$

Let $\eta = y - y_c^j$, where y_c^j is the jth critical level, i.e. $\bar{U}(y_c^j) = c$; then as $\eta \to \pm 0$,

$$\bar{v}_1 \to a_j^{\pm}\phi_a + b_j^{\pm}[\phi_b + p_j\phi_a \log|\eta|] \,, \tag{2.11}$$

where

$$\phi_a = \eta + \tfrac{1}{2}p_j\eta^2 + \dots \,, \quad \phi_b = 1 + q_j\eta^2 + \dots \,,$$

$$p_j = \frac{\bar{U}_{yy}}{\bar{U}_y} \,, \quad q_j = \tfrac{1}{2}\bar{\alpha}^2 + \frac{1}{2}\frac{\bar{U}_{yyy}}{\bar{U}_y} - \frac{\bar{U}_{yy}^2}{\bar{U}_y^2} \,,$$

and $\bar{\alpha} = (\alpha^2 + \beta^2)^{1/2}$. From here up to §4, all basic-flow quantities, such as $\bar{U}_y$, $\bar{U}_{yy}$, $\bar{U}_\tau$, etc., are evaluated at the time τ_0 and at the critical level y_c^j; the subscripts denote the partial derivatives with respect to the variables indicated.

The leading-order solutions for u_1, w_1 and p_1 take the following form:

$$u_1 = A(t_1)\bar{u}_1(y)E\cos\beta z + \bar{u}_1^{(0,2)}(y, t_1)\cos 2\beta z + \text{c.c.} \,,$$

$$w_1 = A(t_1)\bar{w}_1(y)E\sin\beta z + \text{c.c.} \,,$$

$$p_1 = A(t_1)\bar{p}_1(y)E\cos\beta z + \text{c.c.} \,,$$

where as first noted by Goldstein & Choi (1989), a spanwise-dependent mean-flow distortion must be included in u_1 (and also in (2.12) below). The reason for this was further explained in Wu (1992) and Wu *et al.* (1993). We find that as $\eta \to \pm 0$,

$$\bar{u}_1 \to -(\mathrm{i}\alpha)^{-1}\sin^2\theta b_j^{\pm}\eta^{-1} + \dots, \quad \bar{w}_1 \to \bar{\alpha}^{-1}\sin\theta b_j^{\pm}\eta^{-1} + \dots,$$

$$\bar{p}_1 \to \bar{\alpha}^{-1}\bar{U}_y\cos\theta b_j^{\pm} + \dots,$$

where have defined

$$\theta = \tan^{-1}\beta/\alpha .$$

The $O(\epsilon^{4/3})$ term in (2.5), v_2, has the form

$$v_2 = B(t_1)\phi_2(y)E^2 + \bar{v}_2(y,t_1)\cos\beta z E + v^{(0,2)}(y,t_1)\cos\beta z + \text{c.c.} + \dots, \tag{2.12}$$

where $B\phi_2E^2$ is the fundamental planar mode with the scaled amplitude function $B(t_1)$. The function ϕ_2 satisfies Rayleigh's equation. Because it is assumed that $2\alpha = \bar{\alpha}$, we have

$$\phi_2 = \bar{v}_1 . \tag{2.13}$$

The function $\bar{v}_2$ satisfies an inhomogeneous Rayleigh equation. As $\eta \to \pm 0$,

$$\bar{v}_2 \to -b_j^{\pm}r_j\log|\eta| + (a_j^{\pm}r_j + b_j^{\pm}s_j)\eta\log|\eta| + \dots + c_j^{\pm}\phi_a + d_j^{\pm}[\phi_b + p_j\phi_a\log|\eta|] ,$$

where

$$r_j = (\mathrm{i}\alpha)^{-1}\frac{\bar{U}_{yy}}{\bar{U}_y^2}[-\frac{\mathrm{d}A}{\mathrm{d}t_1} - (\mathrm{i}\alpha\bar{U}_\tau\tau_1)A] ,$$

$$s_j = (\mathrm{i}\alpha)^{-1}\left\{(-\mathrm{i}\alpha\bar{U}_{y\tau}\tau_1A)\frac{\bar{U}_{yy}}{\bar{U}_y^2} + (\mathrm{i}\alpha\tau_1A)\frac{\bar{U}_{yy\tau}}{\bar{U}_y} + \left(-\frac{\mathrm{d}A}{\mathrm{d}t_1} - \mathrm{i}\alpha\bar{U}_\tau\tau_1A\right)\frac{\bar{U}_y\bar{U}_{yyy} - \bar{U}_{yy}^2}{\bar{U}_y^3}\right\}.$$

The solvability for $\bar{v}_2$ leads to

$$\mathrm{i}\alpha^{-1}J_1\frac{\mathrm{d}A}{\mathrm{d}t_1} + J_2\tau_1A = -\sum_j\Big[(b_j^+c_j^+ - b_j^-c_j^-) - r_j(b_j^+a_j^+ - b_j^-a_j^-)$$
$$-p_j(b_j^+d_j^+ - b_j^-d_j^-) - (a_j^+d_j^+ - a_j^-d_j^-)\Big] , \tag{2.14}$$

where the sum is over all critical layers, J_1 and J_2 are constants whose definitions can be found in Wu (1992).

At $O(\epsilon^{5/3})$ (see (2.5)), it is sufficient to solve for the deviation of the planar wave eigenfunction from its neutral state; hence we write

$$v_3 = \phi_3(y,t_1)E^2 + \text{c.c.} + \dots . \tag{2.15}$$

The function ϕ_3 satisfies an inhomogeneous Rayleigh equation, and as $\eta \to \pm 0$,

$$\phi_3 \to -b_j^{\pm}R_j\log|\eta| + (a_j^{\pm}R_j + b_j^{\pm}S_j)\eta\log|\eta| + \dots + C_j^{\pm}\phi_a + D_j^{\pm}[\phi_b + p_j\phi_a\log|\eta|] ,$$

where the expressions for R_j and S_j are the same as for r_j and s_j provided that α and A are replaced by 2α and B respectively. The solvability condition for ϕ_3 gives

$$\mathrm{i}(2\alpha)^{-1}J_1\frac{\mathrm{d}B}{\mathrm{d}t_1} + J_2\tau_1B = -\sum_j\Big[(b_j^+C_j^+ - b_j^-C_j^-) - R_j(b_j^+a_j^+ - b_j^-a_j^-)$$
$$-p_j(b_j^+D_j^+ - b_j^-D_j^-) - (a_j^+D_j^+ - a_j^-D_j^-)\Big] . \tag{2.16}$$

The details of derivation of (2.14) and (2.16) can be found in Wu (1992). The amplitude equations will follow from (2.14) and (2.16) after the jumps $(a_j^+ - a_j^-)$, etc. are determined in the next section. The reader who is not particularly interested in the algebraic details can omit the following section at a first reading.

3. Inner expansion

Within the jth critical layer, the appropriate local transverse coordinate is

$$Y = \eta/\epsilon^{1/3} ,$$

and the expansion takes the following form

$$u = \epsilon^{2/3}U_1 + \epsilon^{3/3}U_2 + \epsilon^{4/3}U_3 + +\epsilon^{5/3}U_4 + \dots , \tag{3.1}$$

$$v = \epsilon^{3/3}V_1 + \epsilon^{4/3}V_2 + \epsilon^{5/3}V_3 + \epsilon^{6/3}V_4 + + \dots , \tag{3.2}$$

$$w = \epsilon^{2/3}W_1 + \epsilon^{3/3}W_2 + \epsilon^{4/3}W_3 + \epsilon^{5/3}U_4 + + \dots , \tag{3.3}$$

$$p = \epsilon^{3/3}P_1 + \epsilon^{4/3}P_2 + \epsilon^{5/3}P_3 + \dots . \tag{3.4}$$

The solutions for V_1 and P_1 are just a trivial continuation of the outer expansion, namely

$$V_1 = \hat{A}(t_1)E\cos\beta z + \text{c.c.} , \quad P_1 = \mathrm{i}\alpha^{-1}\bar{U}_y\cos\theta\hat{A}E\cos\beta z + \text{c.c.} , \tag{3.5}$$

where $\hat{A} = b_j A$, and $b_j^+ = b_j^- = b_j$.

Let $W_1 = \hat{W}_1 E\sin\beta z + \text{c.c.}$; then it follows from the z-momentum equation that

$$\hat{L}_0^{(1)}\hat{W}_1 = \mathrm{i}\bar{U}_y\sin\theta\cos\theta\hat{A} , \tag{3.6}$$

where

$$\hat{L}_0^{(n)} = \frac{\partial}{\partial t_1} + n\mathrm{i}\alpha(\bar{U}_y Y + \bar{U}_\tau\tau_1) - \lambda\frac{\partial^2}{\partial Y^2} . \tag{3.7}$$

We solve (3.6) using Fourier transforms, and obtain

$$\hat{W}_1 = \mathrm{i}\bar{U}_y\sin\theta\cos\theta\hat{W}_0^{(0)} , \tag{3.8}$$

where we have defined

$$\hat{W}_0^{(n)} = \int_0^{+\infty}\xi^n\hat{A}(t_1-\xi)\mathrm{e}^{-s\xi^3-\mathrm{i}\Omega\xi}\mathrm{d}\xi , \tag{3.9}$$

and

$$\Omega = \alpha(\bar{U}_y Y + \bar{U}_\tau\tau_1) , \qquad s = \tfrac{1}{3}\lambda\alpha^2\bar{U}_y^2 . \tag{3.10}$$

The leading-order streamwise velocity U_1 can be written as $U_1 = \hat{U}_1 E\cos\beta z + \text{c.c.}$ It follows from the continuity equation that

$$\hat{U}_1 = -\bar{U}_y\sin^2\theta\hat{W}_0^{(0)} . \tag{3.11}$$

At $O(\epsilon^{4/3})$, the pressure P_2 has the solution

$$P_2 = \tfrac{1}{2}\mathrm{i}\alpha^{-1}\bar{U}_y\hat{B}E^2 + \hat{P}_2^{(2,1)}E\cos\beta z + \text{c.c.} , \tag{3.12}$$

where the first term on the right-hand side is the pressure associated with the planar mode; the second term will not be needed in the following analysis.

The vertical velocity at $O(\epsilon^{4/3})$, V_2, satisfies

$$L_0 V_{2,YY} = L_1 V_1 + \frac{\partial}{\partial Y}\left[\frac{\partial S_{11}}{\partial x} + \frac{\partial S_{31}}{\partial z}\right] , \tag{3.13}$$

where

$$L_0 = \frac{\partial}{\partial t_1} + (\bar{U}_y Y + \bar{U}_\tau \tau_1)\frac{\partial}{\partial x} - \lambda \frac{\partial^2}{\partial Y^2} , \tag{3.14}$$

$$L_1 = -\left\{(\tfrac{1}{2}\bar{U}_{yy}Y^2 + \bar{U}_{y\tau}\tau_1 Y + \tfrac{1}{2}\bar{U}_{\tau\tau}\tau_1^2)\tfrac{\partial}{\partial x} + \lambda \tfrac{\partial}{\partial \tau_1}\right\} \frac{\partial^2}{\partial Y^2} + \bar{U}_{yy}\frac{\partial}{\partial x} . \tag{3.15}$$

The Reynolds stresses, S_{11} and S_{31}, are found to be

$$S_{11} = S_{11}^{(0,0)} + S_{11}^{(0,2)} \cos 2\beta z + S_{11}^{(2,0)} E^2 + S_{11}^{(2,2)} E^2 \cos 2\beta z + \text{c.c.} , \tag{3.16}$$

$$S_{31} = S_{31}^{(0,2)} \sin 2\beta z + S_{31}^{(2,2)} E^2 \sin 2\beta z + \text{c.c.} , \tag{3.17}$$

where

$$S_{11}^{(0,0)} = \tfrac{1}{2}\mathrm{i}\alpha \bar{U}_y^2 \sin^2\theta \hat{A}^* \hat{W}_0^{(1)} , \tag{3.18}$$

$$S_{11}^{(0,2)} = \tfrac{1}{2}\mathrm{i}\alpha \bar{U}_y^2 \sin^2\theta \hat{A}^* \hat{W}_0^{(1)} , \tag{3.19}$$

$$S_{11}^{(2,0)} = \tfrac{1}{2}\mathrm{i}\alpha \bar{U}_y^2 \sin^2\theta [\hat{A} \hat{W}_0^{(1)} + 2\sin^2\theta \hat{W}_0^{(0)} \hat{W}_0^{(0)}] , \tag{3.20}$$

$$S_{11}^{(2,2)} = \tfrac{1}{2}\mathrm{i}\alpha \bar{U}_y^2 \sin^2\theta \hat{A} \hat{W}_0^{(1)} , \tag{3.21}$$

$$S_{31}^{(0,2)} = \tfrac{1}{2}\beta \bar{U}_y^2 \cos^2\theta [\hat{A}^* \hat{W}_0^{(1)} + 2\sin^2\theta \hat{W}_0^{(0)} \hat{W}_0^{*(0)}] , \tag{3.22}$$

$$S_{31}^{(2,2)} = \tfrac{1}{2}\beta \bar{U}_y^2 \cos^2\theta \hat{A} \hat{W}_0^{(1)} . \tag{3.23}$$

Equations (3.13), (3.16) and (3.17) indicate that V_2 has a solution of the form

$$V_2 = \hat{V}_2^{(1)} E \cos\beta z + \hat{V}_2^{(0,2)} \cos 2\beta z + \hat{V}_2^{(2,0)} E^2 + \text{c.c.} \tag{3.24}$$

The component $\hat{V}_2^{(1)}$ is driven by the linear forcing term, i.e. $L_1 V_1 = \mathrm{i}\alpha \bar{U}_{yy} \hat{A}$, which is exactly the same as in the two-dimensional case (e.g. see Wu 1991; Wu & Cowley 1994). By analogy, we obtain the jump conditions

$$a_j^+ - a_j^- = \pi \mathrm{i} p_j b_j \mathrm{sgn}(\bar{U}_y) , \tag{3.25}$$

$$d_j^+ - d_j^- = -\pi \mathrm{i} r_j b_j \mathrm{sgn}(\bar{U}_y) . \tag{3.26}$$

Substituting (3.16)–(3.23) into (3.13), we find that $\hat{V}_2^{(0,2)}$ and $\hat{V}_2^{(2,0)}$ satisfy

$$\left\{\frac{\partial}{\partial t_1} - \lambda \frac{\partial^2}{\partial Y^2}\right\} \hat{V}_{2,YY}^{(0,2)} = -\mathrm{i}\bar{S}^3 \sin^2\theta [\hat{A}^* \hat{W}_0^{(2)} + 4\sin^2\theta \hat{W}_0^{*(0)} \hat{W}_0^{(1)}] , \tag{3.27}$$

$$\hat{L}_0^{(2)} \hat{V}_{2,YY}^{(2,0)} = \mathrm{i}\bar{S}^3 \sin^2\theta [\hat{A} \hat{W}_0^{(2)} + 4\sin^2\theta \hat{W}_0^{(0)} \hat{W}_0^{(1)}] , \tag{3.28}$$

where we have put

$$\bar{S} = \alpha \bar{U}_y .$$

The solutions are as follows:

$$\hat{V}_{2,YY}^{(0,2)} = -\mathrm{i}\bar{S}^3 \sin^2\theta \int_0^{+\infty}\int_0^{+\infty} \xi I_v^{(0,2)}(\xi,\eta) \hat{A}^*(t_1-\eta) \hat{A}(t_1-\eta-\xi) \mathrm{e}^{-\mathrm{i}\Omega\xi} \mathrm{d}\xi \mathrm{d}\eta , \tag{3.29}$$

$$\hat{V}_{2,YY}^{(2,0)} = \mathrm{i}\bar{S}^3 \sin^2\theta \int_0^{+\infty}\int_0^{+\infty} I_v^{(2,0)}(\xi,\eta) \hat{A}(t_1-\eta) \hat{A}(t_1-\eta-\xi) \mathrm{e}^{-\mathrm{i}\Omega(\xi+2\eta)} \mathrm{d}\xi \mathrm{d}\eta , \tag{3.30}$$

where

$$I_v^{(0,2)}(\xi,\eta) = I^{(0)}(\xi,\eta)\left[\xi + 4\sin^2\theta \int_0^\eta e^{-2s\zeta^3-3s\xi\zeta^2}d\zeta\right] , \qquad (3.31)$$

$$I_v^{(2,0)}(\xi,\eta) = I^{(1)}(\xi,\eta)\left[\xi^2 + 4\sin^2\theta \int_0^\eta (\xi+2\zeta)e^{2s\zeta^3+3s\xi\zeta^2}d\zeta\right] , \qquad (3.32)$$

$$I^{(0)}(\xi,\eta) = e^{-s(\xi^3+3\xi^2\eta)} , \qquad (3.33)$$

$$I^{(1)}(\xi,\eta) = e^{-s(\xi^3+3\xi^2\eta+6\xi\eta^2+4\eta^3)} . \qquad (3.34)$$

The $O(\epsilon)$ spanwise velocity W_2 (see (3.3)) is governed by

$$L_0 W_2 = -\frac{\partial P_2}{\partial z} - F_2(Y)\frac{\partial W_1}{\partial x} - S_{31} , \qquad (3.35)$$

where

$$F_2(Y) = \tfrac{1}{2}\bar{U}_{yy}Y^2 + \bar{U}_{y\tau}\tau_1 Y + \tfrac{1}{2}\bar{U}_{\tau\tau}\tau_1^2 . \qquad (3.36)$$

The solution for W_2 has the form

$$W_2 = \hat{W}_2^{(1)}E\sin\beta z + \hat{W}_2^{(0,2)}\sin 2\beta z + \hat{W}_2^{(2,2)}E^2\sin 2\beta z + \text{c.c.} \qquad (3.37)$$

As in Wu (1992), for the purpose of deriving the amplitude equations it is sufficient to solve for $\hat{W}_2^{(0,2)}$ and $\hat{W}_2^{(2,2)}$ only. The solution for $\hat{W}_2^{(0,2)}$ follows from the continuity equation, namely

$$\hat{W}_2^{(0,2)} = -(2\beta)^{-1}\hat{V}_{2,Y}^{(0,2)} . \qquad (3.38)$$

The function $\hat{W}_2^{(2,2)}$ satisfies

$$\hat{L}_0^{(2)}\hat{W}_2^{(2,2)} = -S_{31}^{(2,2)} , \qquad (3.39)$$

which is solved to give

$$\hat{W}_2^{(2,2)} = -\tfrac{1}{2}\beta^{-1}\bar{S}^2\sin^2\theta\int_0^{+\infty}\int_0^{+\infty}\xi I^{(1)}(\xi,\eta)\hat{A}(t_1-\eta)\hat{A}(t_1-\eta-\xi)e^{-i\Omega(\xi+2\eta)}d\xi d\eta . \qquad (3.40)$$

The $O(\epsilon)$ streamwise velocity U_2 (see (3.1)) satisfies

$$L_0 U_{2,Y} = -\bar{U}_{yy}V_1 + F_1(Y)\frac{\partial W_1}{\partial z} - F_2(Y)\frac{\partial^2 U_1}{\partial x\partial Y} - \frac{\partial}{\partial Y}S_{11} + \bar{U}_y\frac{\partial W_2}{\partial z} , \qquad (3.41)$$

where

$$F_1(Y) = \bar{U}_{yy}Y + \bar{U}_{Y\tau}\tau_1 . \qquad (3.42)$$

The solution takes the form

$$U_2 = \hat{U}_2^{(1)}E\cos\beta z + \hat{U}_2^{(0,0)} + \hat{U}_2^{(0,2)}\cos 2\beta z + \hat{U}_2^{(2,0)}E^2 + \hat{U}_2^{(2,2)}E^2\cos 2\beta z + \text{c.c.} \qquad (3.43)$$

From the continuity equation, we obtain

$$\hat{U}_2^{(2,0)} = -(2i\alpha)^{-1}\hat{V}_{2,Y}^{(2,0)} , \quad \hat{U}_2^{(2,2)} = i\beta\alpha^{-1}\hat{W}_2^{(2,2)} . \qquad (3.44)$$

The mean-flow distortions $\hat{U}_2^{(0,0)}$ and $\hat{U}_2^{(0,2)}$ are governed by the following equations respectively:

$$\left[\frac{\partial}{\partial t_1} - \lambda\frac{\partial^2}{\partial Y^2}\right]\hat{U}_{2,Y}^{(0,0)} = -\tfrac{1}{2}\alpha^{-1}\bar{S}^3\sin^2\theta\hat{A}^*\hat{W}_0^{(2)} , \qquad (3.45)$$

$$\left[\frac{\partial}{\partial t_1} - \lambda\frac{\partial^2}{\partial Y^2}\right]\hat{U}_{2,Y}^{(0,2)} = -\frac{\partial}{\partial Y}S_{11}^{(0,2)} - \bar{U}_y\hat{V}_{2,Y}^{(0,2)} , \qquad (3.46)$$

where $S_{11}^{(0,2)}$ is defined by (3.19). The solutions are

$$\hat{U}_{2,Y}^{(0,0)} = -\tfrac{1}{2}\alpha^{-1}\bar{S}^3 \sin^2\theta \int_0^{+\infty}\int_0^{+\infty} \xi^2 I^{(0)}(\xi,\eta)\hat{A}^*(t_1-\eta)\hat{A}(t_1-\eta-\xi)\mathrm{e}^{-\mathrm{i}\Omega\xi}\mathrm{d}\xi\mathrm{d}\eta\ , \tag{3.47}$$

$$\hat{U}_{2,Y}^{(0,2)} = -\tfrac{1}{2}\alpha^{-1}\bar{S}^3 \sin^2\theta \int_0^{+\infty}\int_0^{+\infty} I_u^{(0,2)}(\xi,\eta)\hat{A}^*(t_1-\eta)\hat{A}(t_1-\eta-\xi)\mathrm{e}^{-\mathrm{i}\Omega\xi}\mathrm{d}\xi\mathrm{d}\eta\ , \tag{3.48}$$

where we have put

$$I_u^{(0,2)}(\xi,\eta) = I^{(0)}(\xi,\eta)[\xi^2+2\xi\eta+4\sin^2\theta \int_0^{\eta} 2(\eta-\zeta)\mathrm{e}^{-2s\zeta^3-3s\xi\zeta^2}\mathrm{d}\zeta]\ . \tag{3.49}$$

We now seek the solution for V_3; this is found to satisfy

$$L_0 V_{3,YY} = L_1 V_2 + L_2 V_1 + \frac{\partial}{\partial Y}\left[\frac{\partial}{\partial x}S_{12} + \frac{\partial}{\partial z}S_{32}\right]\ , \tag{3.50}$$

where L_1 is defined by (3.15), and

$$\begin{aligned} L_2 = &-[\tfrac{1}{6}\bar{U}_{yyy}Y^3 + \bar{U}_{yy\tau}\tau_1 Y^2 + \bar{U}_{y\tau\tau}\tau_1^2 Y + \tfrac{1}{6}\bar{U}_{\tau\tau\tau}\tau_1^3]\tfrac{\partial^3}{\partial x \partial Y^2} \\ &+[\bar{U}_{yyy}Y + \bar{U}_{yy\tau}\tau_1]\frac{\partial}{\partial x} - \left[\frac{\partial}{\partial t_1} + (\bar{U}_y Y + \bar{U}_\tau \tau_1)\frac{\partial}{\partial x}\right]\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right]\ . \end{aligned} \tag{3.51}$$

The Reynolds stresses S_{12} and S_{32} contain different components, among which only those proportional to $E\cos\beta z$ and $E^3\cos\beta z$ will be needed later. Therefore we write

$$S_{12} = S_{12}^{(1,1)}E\cos\beta z + S_{12}^{(3,1)}E^3\cos\beta z + \text{c.c.} + \dots\ , \tag{3.52}$$

$$S_{32} = S_{32}^{(1,1)}E\sin\beta z + S_{32}^{(3,1)}E^3\sin\beta z + \text{c.c.} + \dots\ . \tag{3.53}$$

After some calculation, we obtain

$$\begin{aligned} S_{12}^{(1,1)} = &\hat{A}\hat{U}_{2,Y}^{*(0,0)} + \tfrac{1}{2}\hat{A}\hat{U}_{2,Y}^{*(0,2)} - \beta\hat{W}_1\hat{U}_2^{*(0,0)} + \tfrac{1}{4}\hat{U}_1\hat{V}_{2,Y}^{*(0,2)} \\ &+[\tfrac{3}{2}\mathrm{i}\alpha\hat{U}_1\hat{U}_2^{*(0,2)} + \tfrac{1}{2}\hat{U}_{1,Y}\hat{V}_2^{*(0,2)}] + [\hat{U}_{1,Y}^*\hat{V}_2^{(2,0)} - \tfrac{1}{2}\hat{U}_1^*\hat{V}_{2,Y}^{(2,0)}]\ , \end{aligned} \tag{3.54}$$

$$S_{12}^{(3,1)} = \hat{A}\hat{U}_{2,Y}^{(2,0)} + \tfrac{1}{2}\hat{A}\hat{U}_{2,Y}^{(2,2)} - 3\beta\hat{U}_1\hat{W}_2^{(2,2)} + [\hat{U}_{1,Y}\hat{V}_2^{(2,0)} - \tfrac{3}{2}\hat{U}_1\hat{V}_{2,Y}^{(2,0)}]\ , \tag{3.55}$$

$$\begin{aligned} S_{32}^{(1,1)} = &[\tfrac{1}{2}\hat{A}\hat{W}_{2,Y}^{*(0,2)} + \mathrm{i}\alpha\hat{W}_1\hat{U}_2^{*(0,0)} - \tfrac{1}{2}\beta\hat{W}_1\hat{W}_2^{*(0,2)}] \\ &+[-\tfrac{1}{2}\mathrm{i}\alpha\hat{W}_1\hat{U}_2^{*(0,2)} - \tfrac{1}{2}\hat{W}_{1,Y}\hat{V}_2^{*(0,2)}] + [\tfrac{1}{2}\hat{W}_1^*\hat{V}_{2,Y}^{(2,0)} + \hat{W}_{1,Y}^*\hat{V}_2^{(2,0)}]\ , \end{aligned} \tag{3.56}$$

$$S_{32}^{(3,1)} = \tfrac{1}{2}\hat{A}\hat{W}_{2,Y}^{(2,2)} - \beta\hat{W}_1\hat{W}_2^{(2,2)} - \tfrac{1}{2}\hat{W}_1\hat{V}_{2,Y}^{(2,0)} + \hat{W}_{1,Y}\hat{V}_2^{(2,0)}\ . \tag{3.57}$$

The derivatives of the Reynolds stresses in (3.50) are found as follows

$$\frac{\partial}{\partial x}S_{12} + \frac{\partial}{\partial z}S_{32} = \bar{M}E\cos\beta z + \tilde{M}E^3\cos\beta z + \hat{M}^{(2,0)}E^2 + \text{c.c.} + \dots\ , \tag{3.58}$$

where

$$\begin{aligned} \bar{M} = &\mathrm{i}\alpha\hat{A}\hat{U}_{2,Y}^{*(0,0)} + \tfrac{1}{2}\mathrm{i}\alpha\hat{A}\hat{U}_{2,Y}^{*(0,2)} + \tfrac{1}{2}\beta\hat{A}\hat{W}_{2,Y}^{*(0,2)} \\ &+\mathrm{i}\alpha\hat{U}_{1,Y}\hat{V}_2^{*(0,2)} - 2\alpha^2\hat{U}_1\hat{U}_2^{*(0,2)} + 2\mathrm{i}\alpha\hat{U}_{1,Y}^*\hat{V}_2^{(2,0)} + \dots\ , \end{aligned} \tag{3.59}$$

$$\bar{M} = -\tfrac{3}{2}\hat{A}\hat{V}^{(2,0)}_{2,YY} - \beta\hat{A}\hat{W}^{(2,2)}_{2,Y} + 8\beta^2\hat{W}_1\hat{W}^{(2,2)}_2 + 4\beta\hat{W}_1\hat{V}^{(2,0)}_{2,Y} - 2\beta\hat{W}_{1,Y}\hat{V}^{(2,0)}_2 \,. \quad (3.60)$$

Here the forcing terms which do not contribute to the jumps have not been included. Equations (3.50) and (3.58) suggest that V_3 has the solution

$$V_3 = \hat{V}_3 E\cos\beta z + \tilde{V}_3 E^3\cos\beta z + \hat{V}^{(2,0)}_3 E^2 + \text{c.c.} + \ldots \,, \quad (3.61)$$

where the component proportional to $E\cos 3\beta z$ has been omitted since it will not be used in deriving the evolution equations. As will be shown below, matching the solution $\hat{V}_3$ will determine $(c_j^+ - c_j^-)$.

The relevant part of the linear forcing term in (3.50), i.e. $(L_1V_2 + L_2V_1)$, is found to be the same as $F^{(l)}(Y,t_1)E$ in Wu & Cowley (1994). Thus the solution forced by it, denoted here by $\hat{V}^{(l)}_3 E$, has the same asymptotic behaviour:

$$\hat{V}^{(l)}_{3,Y} \sim (a_j^+p_j + 2q_jb_j + \tfrac{1}{2}p_j^2b_j)Y + (a_j^+r_j + p_jd_j^+ + s_jb_j)\log|Y| \\ +\{\pm\tfrac{1}{2}\pi\,\mathrm{isgn}(\bar{U}_y)(a_j^+r_j + p_jd_j^+ + s_jb_j) + \ldots\} \,. \quad (3.62)$$

Let $\hat{V}^{(n)}_3$ denote the solution driven by $\bar{M}$ (see (3.50) and (3.58)), i.e.

$$\hat{L}^{(1)}_0\hat{V}^{(n)}_{3,YY} = \bar{M}_Y \,, \quad (3.63)$$

where $\hat{L}^{(1)}_0$ is defined by (3.7); then

$$\hat{V}_{3,YY} = \hat{V}^{(l)}_{3,YY} + \hat{V}^{(n)}_{3,YY} \,. \quad (3.64)$$

We note that because of the form of $\bar{M}_Y$, it is very inconvenient to solve $\hat{V}^{(n)}_{3,Y}$ from (3.63) directly. Moreover, the form of the solution so obtained does not let us evaluate the asymptotic behaviour of $\hat{V}^{(n)}_{3,Y}$ in a straightforward manner. In order to overcome this technical difficulty, we write

$$\hat{V}^{(n)}_{3,YY} = \hat{Q}_v(Y,t_1) - \mathrm{i}\alpha\bar{U}_y^{-1}\hat{U}_{1,YY}\hat{U}^{*(0,2)}_2 + \bar{U}_y^{-1}\hat{U}^*_{1,YY}\hat{V}^{(2,0)}_{2,Y} \,. \quad (3.65)$$

The same method was used in Wu *et al.* (1993), and a similar procedure will be used in solving for $\tilde{V}_3$, $\hat{W}_3$, and $\tilde{W}_3$ later. Substituting (3.65) into (3.63), we find that $\hat{Q}_v(Y,t_1)$ satisfies

$$\hat{L}^{(1)}_0\hat{Q}_v = \hat{M}_v(Y,t_1) \,, \quad (3.66)$$

where

$$\hat{M}_v = \bar{M}_Y + \mathrm{i}\alpha\bar{U}_y^{-1}\hat{L}^{(1)}_0(\hat{U}_{1,YY}\hat{U}^{(0,2)}_2) - \bar{U}_y^{-1}\hat{L}^{(1)}_0(\hat{U}^*_{1,YY}\hat{V}^{(2,0)}_{2,Y}) \,.$$

We note that

$$\hat{L}^{(1)}_0\hat{U}_{1,YY} = -2\mathrm{i}\alpha\bar{U}_y\hat{U}_{1,Y} \,, \quad (3.67)$$

and

$$\hat{L}^{(2)}_0\hat{V}^{(2,0)}_{2,Y} = -2\mathrm{i}\alpha\bar{U}_y\hat{B} + 2\mathrm{i}\alpha\bar{U}_y\hat{V}^{(2,0)}_2 + 2\mathrm{i}\alpha S^{(2,0)}_{11} \,. \quad (3.68)$$

Making use of (3.67), (3.68) and the complex conjugate of (3.46), we find that

$$\hat{M}_v = 2\mathrm{i}\alpha\hat{B}\hat{U}^*_{1,YY} + \mathrm{i}\alpha\hat{A}\hat{U}^{*(0,0)}_{2,YY} + \tfrac{1}{2}\mathrm{i}\alpha\hat{A}\hat{U}^{*(0,2)}_{2,YY} + \tfrac{1}{2}\beta\hat{A}\hat{W}^{*(0,2)}_{2,YY} \\ +\mathrm{i}\alpha\hat{U}_{1,Y}\hat{V}^{*(0,2)}_{2,Y} - 2\alpha^2\hat{U}_1\hat{U}^{*(0,2)}_{2,Y} - 2\mathrm{i}\lambda\alpha\bar{U}_y^{-1}\hat{U}_{1,YYY}\hat{U}^{*(0,2)}_{2,Y} - \mathrm{i}\alpha\bar{U}_y^{-1}\hat{U}_{1,YY}S^{*(0,2)}_{11} \\ -2\mathrm{i}\alpha\bar{U}_y^{-1}\hat{U}^*_{1,YY}S^{(2,0)}_{11} + 2\lambda\bar{U}_y^{-1}\hat{U}^*_{1,YYY}\hat{V}^{(2,0)}_{2,YY} \,, \quad (3.69)$$

where $S^{(0,2)}_{11}$ and $S^{(2,0)}_{11}$ are defined by (3.19) and (3.20) respectively. After solving for $\hat{Q}_v(Y,t_1)$ from (3.66) using Fourier transforms, we obtain

$$\hat{Q}_v = \hat{V}^{(b)}_{3,YY} + \hat{V}^{(r)}_{3,YY} \,, \quad (3.70)$$

where

$$\hat{V}^{(b)}_{3,YY} = 2\mathrm{i}\bar{S}^3 \sin^2\theta \int_0^{+\infty}\int_0^{+\infty} \xi^2 \mathrm{e}^{-2s\xi^3 - s(\eta-\xi)^3} \hat{A}^*(t_1-\xi-\eta)\hat{B}(t_1-\eta)\mathrm{e}^{-\mathrm{i}\Omega(\eta-\xi)}\mathrm{d}\xi\mathrm{d}\eta \ , \quad (3.71)$$

$$\hat{V}^{(r)}_{3,YY} = \bar{S}^4 \sin^2\theta \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \hat{K}_v(\xi,\eta,\zeta)\mathrm{e}^{-\mathrm{i}\Omega(\zeta-\xi)} \times \hat{A}(t_1-\zeta)\hat{A}(t_1-\zeta-\eta)\hat{A}^*(t_1-\zeta-\eta-\xi)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta \ . \quad (3.72)$$

The kernel $\hat{K}_v(\xi,\eta,\zeta)$ is quite complicated and is given in Appendix A. Matching $\hat{V}_{3,Y}$ with the outer expansion, we find that

$$c_j^+ - c_j^- = \hat{V}_{3,Y}(+\infty) - \hat{V}_{3,Y}(-\infty) \ . \quad (3.73)$$

After making use of (3.62), (3.64), (3.65), (3.70)-(3.73), we obtain

$$c_j^+ - c_j^- = \pi \mathrm{i}\,\mathrm{sgn}(\bar{U}_y)(a_j^+ r_j + p_j d_j^+ + s_j b_j) + 4\pi\mathrm{i}\alpha^2 \bar{U}_y|\bar{U}_y| \sin^2\theta \int_0^{+\infty} \xi^2 \mathrm{e}^{-2s\xi^3}\hat{A}(t_1-2\xi)\hat{B}(t_1-\xi)\mathrm{d}\xi + \pi\alpha^3|\bar{U}_y|^3 \sin^2\theta \int_0^{+\infty}\int_0^{+\infty} K_a(\xi,\eta|\lambda)\hat{A}(t_1-\xi)\hat{A}(t_1-\xi-\eta)\hat{A}^*(t_1-2\xi-\eta)\mathrm{d}\xi\mathrm{d}\eta \ , \quad (3.74)$$

where the kernel function K_a was first derived by Wu *et al.* (1993) for an arbitrary obliqueness angle θ, but for completeness we reproduce it in Appendix B. Although the kernel $K_a(\xi,\eta)$ is algebraically complicated, it simplifies to the following form when $\lambda = 0$ (cf. Wu *et al.* 1993; Wu 1992):

$$K_a(\xi,\eta) = -\tfrac{1}{4}\xi(4\xi^2 + 5\xi\eta + 3\eta^2) \ . \quad (3.75)$$

So far we have obtained the necessary jumps to derive the amplitude equation for the oblique waves. To derive the amplitude equation for the planar wave, we need to seek the jumps $(D_j^+ - D_j^-)$ and $(C_j^+ - C_j^-)$. The jump $(D_j^+ - D_j^-)$ can be obtained by solving for $\hat{V}_3^{(2,0)}$ (see (3.61)); this satisfies the equation

$$\hat{L}_0^{(2)}\hat{V}^{(2,0)}_{3,YY} = 2\mathrm{i}\alpha\bar{U}_{yy}\hat{B} \ . \quad (3.76)$$

After solving for $\hat{V}^{(2,0)}_{3,YY}$ and matching $\hat{V}_3^{(2,0)}$ with the appropriate outer solution, we obtain

$$D_j^+ - D_j^- = -\pi\mathrm{i}R_j b_j \mathrm{sgn}(\bar{U}_y) \ . \quad (3.77)$$

The remaining jump condition to be determined is $(C_j^+ - C_j^-)$. This can be obtained by solving for V_4 (see (3.2)). To this end, we first need to know the harmonic component $\tilde{V}_3$ and W_3.

The harmonic component $\tilde{V}_3$ is governed by

$$\hat{L}_0^{(3)}\tilde{V}_{3,YY} = \tilde{M}_Y \ , \quad (3.78)$$

where $\tilde{M}$ is defined by (3.60). As for $\hat{V}^{(n)}_{3,YY}$ (see (3.65)), to aid the calculation of the asymptotic behaviour, we write

$$\tilde{V}_{3,YY} = \tilde{Q}_v + \mathrm{i}\beta\bar{S}^{-1}\hat{W}_{1,YY}\hat{V}^{(2,0)}_{2,Y} \ . \quad (3.79)$$

We find that $\tilde{Q}_v$ satisfies

$$\hat{L}_0^{(3)}\tilde{Q}_v = \tilde{M}_v\ , \tag{3.80}$$

where

$$\tilde{M}_v = -2\beta\hat{B}\hat{W}_{1,YY} - \tfrac{3}{2}\hat{A}\hat{V}^{(0,2)}_{2,YYY} - \beta\hat{A}\hat{W}^{(2,2)}_{2,YY} + [8\beta^2(\hat{W}_1\hat{W}^{(2,0)}_{2,YY})_Y + 4\beta\hat{W}_1\hat{V}^{(2,0)}_{2,YY}]$$
$$-\mathrm{i}\beta\bar{S}^{-1}(2\mathrm{i}\alpha)\hat{W}_{1,YY}S^{(2,0)}_{11} + 2\lambda(\mathrm{i}\beta\bar{S}^{-1})\hat{W}_{1,YYY}\hat{V}^{(2,0)}_{2,YY}\ . \tag{3.81}$$

Solving (3.80), we obtain

$$\tilde{Q}_v = \tilde{V}^{(r)}_{3,YY} + \tilde{V}^{(b)}_{3,YY}\ , \tag{3.82}$$

where

$$\tilde{V}^{(b)}_{3,YY} = 2\mathrm{i}\bar{S}^3\sin^2\theta\int_0^{+\infty}\int_0^{+\infty}\xi^2\mathrm{e}^{-2s\xi^3/3-s(\xi+3\eta)^3/3}\hat{A}(t_1-\xi-\eta)\hat{B}(t_1-\eta)\mathrm{e}^{-\mathrm{i}\Omega(\xi+3\eta)}\mathrm{d}\xi\mathrm{d}\eta\ , \tag{3.83}$$

$$\tilde{V}^{(r)}_{3,YY} = \bar{S}^4\sin^2\theta\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\tilde{K}_v(\xi,\eta,\zeta)\mathrm{e}^{-\mathrm{i}\Omega(\xi+2\eta+3\zeta)}$$
$$\times\hat{A}(t_1-\zeta)\hat{A}(t_1-\zeta-\eta)\hat{A}(t_1-\zeta-\eta-\xi)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta\ . \tag{3.84}$$

The function for $\tilde{K}_v(\xi,\eta,\zeta)$ is given in Appendix A.

We now seek the solution for W_3. This term satisfies

$$L_0W_3 = -\frac{\partial P_3}{\partial z} - F_2(Y)\frac{\partial W_2}{\partial x} - F_3(Y)\frac{\partial W_1}{\partial x} - S_{32}\ , \tag{3.85}$$

where

$$F_3(Y) = \tfrac{1}{6}\bar{U}_{yyy}Y^3 + \bar{U}_{yy\tau}\tau_1Y^2 + \bar{U}_{y\tau\tau}\tau_1^2Y + \tfrac{1}{6}\bar{U}_{\tau\tau\tau}\tau_1^3\ .$$

We write

$$W_3 = \hat{W}_3E\sin\beta z + \tilde{W}_3E^3\sin\beta z + \text{c.c.} + \dots\ , \tag{3.86}$$

where the components irrelevant to the generation of the two-dimensional fundamental at the next order have not been included. We further note that the solution driven by the first three terms on the right-hand side of (3.85) does not contribute to the jump $(C_j^+ - C_j^-)$. So we shall concentrate on the solution driven by $-S_{32}$, i.e. it is sufficient to solve for $\hat{W}_3$ which satisfies

$$\hat{L}_0^{(1)}\hat{W}_3 = -S^{(1,1)}_{32}\ , \tag{3.87}$$

where $S^{(1,1)}_{32}$ is defined by (3.56). Following a similar procedure to that for solving $\hat{V}_3$, we write

$$\hat{W}_3 = \hat{Q}_w + \tfrac{1}{2}\mathrm{i}\bar{S}^{-1}\hat{W}^*_{1,Y}\hat{V}^{(2,0)}_{2,Y} - \tfrac{1}{2}\bar{U}_y^{-1}\hat{W}_{1,Y}\hat{U}^{*(0,2)}_2\ . \tag{3.88}$$

The function $\hat{Q}_w$ satisfies

$$\hat{L}_0^{(1)}\hat{Q}_w = -\hat{B}\hat{W}^*_{1,Y} - [\tfrac{1}{2}A\hat{W}^{*(0,2)}_{2,Y} + \mathrm{i}\alpha\hat{W}_1\hat{U}^{*(0,0)}_{2,Y} - \tfrac{1}{2}\hat{W}_1\hat{W}^{*(0,2)}_2]$$
$$+[\bar{U}_y^{-1}\hat{W}^*_{1,Y}S^{(2,0)}_{11} + \lambda\mathrm{i}\bar{S}^{-1}\hat{W}^*_{1,YY}\hat{V}^{(2,0)}_{2,YY}]$$
$$+[-\tfrac{1}{2}\bar{U}_y^{-1}\hat{W}_{1,Y}S^{*(0,2)}_{11} - \lambda\bar{U}_y^{-1}\hat{W}_{1,YY}\hat{U}^{*(0,2)}_{2,Y}]\ . \tag{3.89}$$

We find

$$\hat{Q}_w = \hat{W}^{(b)}_3 + \hat{W}^{(r)}_3\ , \tag{3.90}$$

where

$$\hat{W}_3^{(b)} = -\beta^{-1}\bar{S}^2\sin^2\theta\int_0^{+\infty}\int_0^{+\infty}\xi e^{-2s\xi^3-s(\eta-\xi)^3}\hat{A}^*(t_1-\xi-\eta)\hat{B}(t_1-\eta)e^{-i\Omega(\eta-\xi)}d\xi d\eta\ , \quad (3.91)$$

$$\hat{W}_3^{(r)} = -i\beta^{-1}\bar{S}^3\sin^2\theta\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\hat{K}_w(\xi,\eta,\zeta)e^{-i\Omega(\zeta-\xi)} \times\hat{A}(t_1-\zeta)\hat{A}(t_1-\zeta-\eta)\hat{A}^*(t_1-\zeta-\eta-\xi)d\xi d\eta d\zeta\ , \quad (3.92)$$

and the kernel $\hat{K}_w(\xi,\eta,\zeta)$ is given in Appendix A.

The harmonic component $\tilde{W}_3$ (see (3.86)) satisfies the equation

$$\hat{L}_0^{(3)}\tilde{W}_3 = -S_{32}^{(3,1)}\ , \quad (3.93)$$

where $S_{32}^{(3,1)}$ is defined by (3.57). The solution can be written as

$$\tilde{W}_3 = \tilde{Q}_w + \tfrac{1}{2}i\bar{S}^{-1}\hat{W}_{1,Y}\hat{V}_{2,Y}^{(2,0)}\ , \quad (3.94)$$

with $\tilde{Q}_w$ satisfying

$$\hat{L}_0^{(3)}\tilde{Q}_w = -\hat{B}\hat{W}_{1,Y} - [\tfrac{1}{2}\hat{A}\hat{W}_{2,Y}^{(2,2)} - \beta\hat{W}_1\hat{W}_2^{(2,2)}] + \bar{U}_y^{-1}\hat{W}_{1,Y}S_{11}^{(2,0)} + i\lambda\bar{S}^{-1}\hat{W}_{1,YY}\hat{V}_{2,YY}^{(2,0)}\ . \quad (3.95)$$

We find that

$$\tilde{Q}_w = \tilde{W}_3^{(b)} + \tilde{W}_3^{(r)}\ , \quad (3.96)$$

where

$$\tilde{W}_3^{(b)} = -\beta^{-1}\bar{S}^2\sin^2\theta\int_0^{+\infty}\int_0^{+\infty}\xi e^{-2s\xi^3/3-s(\xi+3\eta)^3/3}\hat{A}(t_1-\xi-\eta)\hat{B}(t_1-\eta)e^{-i\Omega(\xi+3\eta)}d\xi d\eta\ , \quad (3.97)$$

$$\tilde{W}_3^{(r)} = -i\beta^{-1}\bar{S}^3\sin^2\theta\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\tilde{K}_w(\xi,\eta,\zeta)e^{-i\Omega(\xi+2\eta+3\zeta)} \times\hat{A}(t_1-\zeta)\hat{A}(t_1-\zeta-\eta)\hat{A}(t_1-\zeta-\eta-\xi)d\xi d\eta d\zeta\ , \quad (3.98)$$

and the kernel $\tilde{K}_w(\xi,\eta,\zeta)$ is given in Appendix A.

The streamwise velocity at $O(\epsilon^{4/3})$ takes the form

$$U_3 = \hat{U}_3 E\cos\beta z + \tilde{U}_3 E^3\cos\beta z + \text{c.c.} + \dots\ . \quad (3.99)$$

It follows from the continuity equation that

$$\hat{U}_3 = -(i\alpha)^{-1}(\hat{V}_{3,Y} + \beta\hat{W}_3)\ , \quad (3.100)$$

$$\tilde{U}_3 = -(3i\alpha)^{-1}(\tilde{V}_{3,Y} + \beta\tilde{W}_3)\ . \quad (3.101)$$

We are now in a position to solve for V_4 (see (3.2)) to determine $(C_j^+ - C_j^-)$. To this end, it suffices to seek only the two-dimensional fundamental component in V_4, i.e.

$$V_4 = \hat{V}_4^{(2,0)}E^2 + \text{c.c.} + \dots\ . \quad (3.102)$$

The function $\hat{V}_4^{(2,0)}$ satisfies the following equation:

$$\hat{L}_0^{(2)}\hat{V}_{4,YY}^{(2,0)} = \hat{L}_1^{(2)}\hat{V}_3^{(2,0)} + \hat{L}_2^{(2)}\hat{V}_2^{(2,0)} + R_4 + \dots\ , \quad (3.103)$$

where again forcing terms which do contribute the jump $(C_j^+ - C_j^-)$ have been ignored.

The nonlinear forcing term R_4 is given in Appendix C†, and

$$\hat{L}_1^{(2)} = -\left\{2i\alpha(\tfrac{1}{2}\bar{U}_{yy}Y^2 + \bar{U}_{y\tau}\tau_1 Y + \tfrac{1}{2}\bar{U}_{\tau\tau}\tau_1^2) + \lambda\tfrac{\partial}{\partial\tau_1}\right\}\frac{\partial^2}{\partial Y^2} + 2i\alpha\bar{U}_{yy}\,,$$

$$\hat{L}_2^{(2)} = -2i\alpha[\tfrac{1}{6}\bar{U}_{yyy}Y^3 + \bar{U}_{yy\tau}\tau_1 Y^2 + \bar{U}_{y\tau\tau}\tau_1^2 Y + \tfrac{1}{6}\bar{U}_{\tau\tau\tau}\tau_1^3]\tfrac{\partial^2}{\partial Y^2}$$
$$+2i\alpha[\bar{U}_{yyy}Y + \bar{U}_{yy\tau}\tau_1] + 4\alpha^2\left[\frac{\partial}{\partial t_1} + 2i\alpha(\bar{U}_y Y + \bar{U}_\tau\tau_1)\right]\,.$$

Let $\hat{V}_4^{(l)}$ and $\hat{V}_4^{(n)}$ denote the solutions driven by $(\hat{L}_1^{(2)}\hat{V}_3^{(2,0)} + \hat{L}_2^{(2)}\hat{V}_2^{(2,0)})$ and by R_4 respectively. Following a similar procedure to that for the two-dimensional case (Wu 1991), we find

$$\hat{V}_{4,Y}^{(l)}(+\infty) - \hat{V}_{4,Y}^{(l)}(-\infty) = \pi i\,\mathrm{sgn}(\bar{U}_y)(a_j^+ R_j + p_j D_j^+ + b_j S_j)\,, \tag{3.104}$$

where the definitions for p_j, R_j and S_j are stated in §2.

The solution for $\hat{V}_{4,YY}^{(n)}$ can be written as

$$\hat{V}_{4,YY}^{(n)} = Q_4 + \hat{V}_{4,YY}^{(s)}\,, \tag{3.105}$$

where

$$\hat{V}_{4,YY}^{(s)} = \tfrac{1}{3}\bar{U}_y^{-1}\hat{U}_{1,YY}^{*}[\tilde{V}_{3,Y} + \beta\tilde{W}_3] + \bar{U}_y^{-1}\hat{U}_{1,YY}[\hat{V}_{3,Y} + \beta\hat{W}_3]$$
$$-\tfrac{1}{2}\bar{U}_y^{-2}(\hat{U}_{1,Y}\hat{U}_{1,Y}^{*})_Y\hat{V}_{2,Y}^{(2,0)} + \bar{U}_y^{-1}\hat{U}_{2,YY}^{*(0,0)}\hat{V}_{2,Y}^{(2,0)}$$
$$+\tfrac{1}{2}i\alpha\bar{U}_y^{-2}\hat{U}_{1,Y}\hat{U}_{1,YY}\hat{U}_2^{*(0,2)} + \beta\bar{U}_y^{-1}\hat{W}_{2,YY}^{(2,2)}\hat{U}_2^{*(0,2)}\,. \tag{3.106}$$

It follows from the expansion of the x-momentum and continuity equations that

$$\hat{L}_0^{(1)}(\hat{V}_{3,Y} + \beta\hat{W}_3) = i\alpha\bar{U}_y\hat{V}_3 + i\alpha S_{1,2}^{(1,1)}\,, \tag{3.107}$$

$$\hat{L}_0^{(3)}(\tilde{V}_{3,Y} + \beta\tilde{W}_3) = 3i\alpha\bar{U}_y\tilde{V}_3 + 3i\alpha S_{1,2}^{(3,1)}\,. \tag{3.108}$$

Using the above relations and (3.68), we find that Q_4 satisfies

$$\hat{L}_0^{(2)}Q_4 = N^{(b)} + N^{(r)}\,, \tag{3.109}$$

where the functions $N^{(b)}$ and $N^{(r)}$ are defined in Appendix C.

In our opinion, the subtraction of $\hat{V}_{4,YY}^{(s)}$ from $\hat{V}_{4,YY}^{(n)}$ and similar subtractions in solving for $\hat{V}_{3,YY}^{(n)}$ and $\tilde{V}_{3,YY}$ (see (3.65) and (3.79)), etc. are a crucial step to circumvent the difficulty of calculating the asymptotic behaviour of $\hat{V}_{4,Y}^{(n)}$. This technique simply makes use of certain features of the critical-layer equations, e.g. (3.46), (3.67) and (3.68). Since these equations are generic, the technique identified here could be useful for other related critical-layer analyses.

Let $\hat{V}_4^{(r)}$ and $\hat{V}_4^{(b)}$ denote the solutions driven by $N^{(r)}$ and $N^{(b)}$ respectively, i.e.

$$\hat{L}_0^{(2)}\hat{V}_{4,YY}^{(b)} = N^{(b)}\,,\quad \hat{L}_0^{(2)}\hat{V}_{4,YY}^{(r)} = N^{(r)}\,; \tag{3.110}$$

then

$$\hat{V}_{4,YY} = \hat{V}_{4,YY}^{(l)} + \hat{V}_{4,YY}^{(r)} + \hat{V}_{4,YY}^{(s)} + \hat{V}_{4,YY}^{(b)}\,. \tag{3.111}$$

† Appendix C is available from the author or the JFM Editorial Office.

Matching $\hat{V}_{4,Y}$ with the appropriate outer expansion, we find

$$C_j^+ - C_j^- = \hat{V}_{4,Y}(+\infty) - \hat{V}_{4,Y}(-\infty) \,. \tag{3.112}$$

We note that if $\Gamma^{(b)}(k)$ and $\Gamma^{(r)}(k)$ are the Fourier transform of $\hat{V}^{(b)}_{4,YY}$ and $\hat{V}^{(r)}_{4,YY}$ respectively, i.e.

$$\Gamma^{(b)}(k) = \int_{-\infty}^{+\infty} \hat{V}^{(b)}_{4,YY} \mathrm{e}^{-\mathrm{i}kY} \mathrm{d}Y \ , \quad \Gamma^{(r)}(k) = \int_{-\infty}^{+\infty} \hat{V}^{(r)}_{4,YY} \mathrm{e}^{-\mathrm{i}kY} \mathrm{d}Y \ ,$$

then

$$\hat{V}^{(b)}_{4,Y}(+\infty) - \hat{V}_{4,Y}(-\infty)^{(b)} = \Gamma^{(b)}(0) \ , \quad \hat{V}^{(r)}_{4,Y}(+\infty) - \hat{V}_{4,Y}(-\infty)^{(r)} = \Gamma^{(r)}(0) \,. \tag{3.113}$$

It is straightforward, though lengthy, to solve for $\Gamma^{(b)}$ and $\Gamma^{(r)}$ by Fourier transforming (3.110). The jump associated with $\hat{V}^{(s)}_{4,YY}$, i.e. $[\hat{V}^{(s)}_{4,Y}(+\infty) - \hat{V}_{4,Y}(-\infty)^{(s)}]$, can be obtained directly from (3.106) by integration by parts. After a tedious calculation, we obtain

$$\begin{aligned} C_j^+ - C_j^- &= \pi \mathrm{i}\,\mathrm{sgn}(\bar{U}_y)(a_j^+ R_j + p_j D_j^+ + b_j S_j) \\ &+ \tfrac{1}{2}\bar{S}^4 \sin^2\theta j_0 \int_0^{+\infty}\!\!\int_0^{+\infty} K_{21}(\xi,\eta)\hat{A}(t_1-\xi)\hat{B}(t_1-\xi-\eta)\hat{A}^*(t_1-3\xi-2\eta)\mathrm{d}\xi\mathrm{d}\eta \\ &+ \tfrac{1}{2}\bar{S}^4 j_0 \sin^2\theta \int_0^{+\infty}\!\!\int_0^{+\infty} K_{22}(\xi,\eta)\hat{B}(t_1-\xi)\hat{A}(t_1-\xi-\eta)\hat{A}^*(t_1-3\xi-\eta)\mathrm{d}\xi\mathrm{d}\eta \\ &+ 2\mathrm{i}\bar{S}^5 \sin^2\theta j_0 \int_0^{+\infty}\!\!\int_0^{+\infty}\!\!\int_0^{+\infty} K_b(\xi,\eta,\zeta)\hat{A}(t_1-\zeta)\hat{A}(t_1-\zeta-\eta) \\ &\qquad \times \hat{A}(t_1-\zeta-\eta-\xi)\hat{A}^*(t_1-3\zeta-2\eta-\xi)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta \ , \end{aligned} \tag{3.114}$$

where $j_0 = \pi|\bar{S}|^{-1}$. The kernels K_{21}, K_{22} and $K_b(\xi,\eta,\zeta)$ are rather complicated and are relegated to Appendices B and D. But in the inviscid limit ($\lambda = 0$), they revert to (cf. Wu 1992)

$$K_{21} = 4\xi(\xi+\eta)(2\xi+\eta) \ , \quad K_{22} = 8\xi^3 \ , \tag{3.115}$$

$$K_b(\xi,\eta,\zeta) = \tfrac{1}{16}[-3\zeta^4 - 4(\xi+2\eta)\zeta^3 - (3\xi^2+\xi\eta+2\eta^2)\zeta^2 + (3\xi^2+5\xi\eta+4\eta^2)\eta\zeta] \,. \tag{3.116}$$

4. Evolution equations for the amplitudes

4.1. *Coupled amplitude equations*

Substituting the jumps (3.25), (3.26) and (3.74) into (2.14), we obtain the following amplitude equation for the oblique waves:

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t_1} &= \kappa_a A + \int_0^{+\infty} \sum g_{11}\xi^2 \mathrm{e}^{-2s\xi^3} A^*(t_1-2\xi)B(t_1-\xi)\mathrm{d}\xi \\ &+ \int_0^{+\infty}\!\!\int_0^{+\infty} \sum g_{12}K_a(\xi,\eta)A(t_1-\xi)A(t_1-\xi-\eta)A^*(t_1-2\xi-\eta)\mathrm{d}\xi\mathrm{d}\eta \ , \end{aligned} \tag{4.1}$$

where we have written $\kappa_a = f_0\tau_1/f$, and $g_{1k} = f_{1k}/f$ ($k = 1,2$). The constants f and f_0 have the same expressions as in the inviscid case (Wu 1992) and will not be repeated here. The constants f_{11} and f_{12} are

$$f_{11} = -4\pi\mathrm{i}\alpha^2 \sin^2\theta b_j |b_j|^2 \bar{U}_y |\bar{U}_y| \ , \quad f_{12} = -\pi\alpha^3 \sin^2\theta b_j^2 |b_j|^2 |\bar{U}_y|^3 \,. \tag{4.2}$$

The kernel $K_a(\xi,\eta)$ is given by (B 1). Equation (4.1) and the first expression in (4.2) indicate that for the resonant triad of sinuous modes in a symmetric shear flow (e.g. in a plane wake), the quadratic term vanishes because the contribution from each critical layer cancels each other.

Similarly, substitution of (3.25), (3.77) and (3.114) into (2.16) gives the amplitude equation for the planar wave, namely

$$\frac{\mathrm{d}B}{\mathrm{d}t_1} = \kappa_b B + \int_0^{+\infty}\int_0^{+\infty} \sum g_{21}K_{21}(\xi,\eta)A(t_1-\xi)B(t_1-\xi-\eta)A^*(t_1-3\xi-2\eta)\mathrm{d}\xi\mathrm{d}\eta$$
$$+\int_0^{+\infty}\int_0^{+\infty} \sum g_{21}K_{22}(\xi,\eta)B(t_1-\xi)A(t_1-\xi-\eta)A^*(t_1-3\xi-\eta)\mathrm{d}\xi\mathrm{d}\eta$$
$$+\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \sum g_{22}K_b(\xi,\eta,\zeta)A(t_1-\zeta)A(t_1-\zeta-\eta)$$
$$\times A(t_1-\zeta-\eta-\xi)A^*(t_1-3\zeta-2\eta-\xi)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta\ , \qquad (4.3)$$

where we have written $\kappa_b = 2f_0\tau_1/f$, $g_{2k} = 2f_{2k}/f$ $(k = 1,2)$, and

$$f_{21} = -\tfrac{1}{2}\pi\alpha^3 \sin^2\theta b_j^2|b_j|^2|\bar{U}_y|^3\ , \quad f_{22} = -2\pi\mathrm{i}\alpha^4\sin^2\theta b_j^3|b_j|^2\bar{U}_y^3|\bar{U}_y|\ . \qquad (4.4)$$

The kernel $K_b(\xi,\eta,\zeta)$ is defined by (D 1) in Appendix D.

In order to match to the earlier linear stage, the amplitudes A and B must have the 'initial' conditions (see e.g. Goldstein & Leib 1988)

$$A \to A_0\mathrm{e}^{\kappa_a t_1}\ , \quad B \to B_0\mathrm{e}^{\kappa_b t_1} \quad \text{as} \quad t_1 \to -\infty\ . \qquad (4.5)$$

We now rescale the amplitude equations by introducing the following variables:

$$\bar{t} = \kappa_{br}t_1 - t_0\ , \quad \bar{\lambda} = \lambda/\kappa_{br}^3\ , \qquad (4.6)$$

$$\bar{A} = A\mathrm{e}^{-\mathrm{i}(T_A+\kappa_{bi}t_1)/2}|\sum g_{12}|^{1/2}/\lambda_A\ , \quad \bar{B} = B\mathrm{e}^{-\mathrm{i}(T_B+\kappa_{bi}t_1)}|\sum g_{11}|/\kappa_{br}^4\ , \qquad (4.7)$$

where κ_{br} and κ_{bi} are the real and imaginary parts of κ_b. The constants t_0, T_A, T_B and λ_A are all real, and are chosen to satisfy

$$\mathrm{e}^{\mathrm{i}T_B}\kappa_{br}^4/|\sum g_{11}| = B_0\mathrm{e}^{t_0}\ , \quad \mathrm{e}^{\mathrm{i}T_A}\lambda_A/|\sum g_{12}|^{1/2} = A_0\mathrm{e}^{\kappa t_0}\ , \quad \kappa = (\kappa_a - \tfrac{1}{2}\mathrm{i}\kappa_{bi})/\kappa_{br}\ .$$

The rescaled amplitude equations and the initial conditions then become

$$\frac{\mathrm{d}\bar{A}}{\mathrm{d}\bar{t}} = \kappa\bar{A} + \mathrm{e}^{-\mathrm{i}\varphi_0}\int_0^{+\infty} \sum \bar{g}_{11}\xi^2\mathrm{e}^{-2s\xi^3}\bar{A}^*(\bar{t}-2\xi)\bar{B}(\bar{t}-\xi)\mathrm{d}\xi$$
$$+\chi_0\int_0^{+\infty}\int_0^{+\infty} \sum \bar{g}_{12}K_a(\xi,\eta)\bar{A}(\bar{t}-\xi)\bar{A}(\bar{t}-\xi-\eta)\bar{A}^*(\bar{t}-2\xi-\eta)\mathrm{d}\xi\mathrm{d}\eta\ , \qquad (4.8)$$

$$\frac{\mathrm{d}\bar{B}}{\mathrm{d}\bar{t}} = \bar{B} + \chi_0\int_0^{+\infty}\int_0^{+\infty} \sum \bar{g}_{21}K_{21}(\xi,\eta)\bar{A}(\bar{t}-\xi)\bar{B}(\bar{t}-\xi-\eta)\bar{A}^*(\bar{t}-3\xi-2\eta)\mathrm{d}\xi\mathrm{d}\eta$$
$$+\chi_0\int_0^{+\infty}\int_0^{+\infty} \sum \bar{g}_{21}K_{22}(\xi,\eta)\bar{B}(\bar{t}-\xi)\bar{A}(\bar{t}-\xi-\eta)\bar{A}^*(\bar{t}-3\xi-\eta)\mathrm{d}\xi\mathrm{d}\eta$$
$$+\chi_0^2\mathrm{e}^{\mathrm{i}\varphi_0}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \sum \bar{g}_{22}K_b(\xi,\eta,\zeta)\bar{A}(\bar{t}-\zeta)\bar{A}(\bar{t}-\zeta-\eta)$$
$$\times \bar{A}(\bar{t}-\zeta-\eta-\xi)\bar{A}^*(\bar{t}-3\zeta-2\eta-\xi)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta\ , \qquad (4.9)$$

$$\bar{A} \to \mathrm{e}^{\kappa\bar{t}}\ , \quad \bar{B} \to \mathrm{e}^{\bar{t}} \quad \text{as} \quad \bar{t} \to -\infty\ , \qquad (4.10)$$

where

$$\bar{g}_{11} = g_{11}/|\sum g_{11}| , \quad \bar{g}_{12} = g_{12}/|\sum g_{12}| ,$$

$$\bar{g}_{21} = g_{21}/|\sum g_{12}| , \quad \bar{g}_{22} = g_{22}|\sum g_{11}|/|\sum g_{22}|^2 ,$$

$$\chi_0 = \kappa_{br}^{8\gamma-6}\left\{|\sum g_{11}|^{-2\gamma}|\sum g_{12}|\right\}|A_0|^2/|B_0|^{2\gamma} , \quad \gamma = \kappa_{ar}/\kappa_{br} , \tag{4.11}$$

$$\varphi_0 = arg[A_0^2/B_0] + \kappa_i . \tag{4.12}$$

In rescaling, we have also made the substitution: $\kappa_{br}\xi \to \xi$, $\kappa_{br}\eta \to \eta$, $\kappa_{br}\zeta \to \zeta$. The kernel functions remain unchanged except that λ is replaced by $\bar{\lambda}$, but for convenience we still write $\bar{\lambda}$ as λ without losing generality. The real parameters φ_0 and χ_0 account for the effects of the initial phase difference and the relative amplitude of the planar and the oblique modes respectively.

4.2. *Effect of initial amplitudes and parametric resonance*

The coupled amplitude equations (4.8) and (4.9) are formally derived when

$$\epsilon = O(\delta^{3/4}) . \tag{4.13}$$

If $\epsilon \ll O(\delta^{3/4})$, i.e. $|\chi_0| \ll 1$, then they reduce to

$$\frac{d\bar{A}}{d\bar{t}} = \kappa\bar{A} + e^{-i\varphi_0}\int_0^{+\infty}\sum g_{11}\xi^2 e^{-2s\xi^3}\bar{A}^*(\bar{t}-2\xi)\bar{B}(\bar{t}-\xi)d\xi , \tag{4.14}$$

$$\frac{d\bar{B}}{d\bar{t}} = \bar{B} . \tag{4.15}$$

These are the evolution equations describing the parametric resonance, in which the oblique modes have no back reaction on the planar mode. Equations (4.14) and (4.15) were first derived by Wundrow, Hultgren & Goldstein (1994). (The inviscid version of (4.14) was first derived by Goldstein & Lee 1992). We note that provided $\delta \geqslant O[(\alpha c_i)^4] \gg \epsilon^{4/3}$, the parametric resonance occurs even when the magnitude of the oblique modes is infinitesimal. Parametric resonance effect is negligible when $\delta \ll (\alpha c_i)^4$. As shown by Goldstein & Lee (1992), in the parametric-resonance regime the three-dimensional waves experience a super-exponential growth, while the two-dimensional wave continues to grow exponentially. Depending on the initial magnitude of the oblique modes and the nature of the critical layers, the subsequent stage which follows the parametric resonance can take different forms. If the initial oblique modes are 'algebraically' small, their magnitude will quickly increase to $O(\delta^{3/4})$ due to the super-exponential growth, and the evolution soon enters the regime described by the fully coupled equations (4.8) and (4.9); in this case the fully coupled equations are uniformly valid in the two regimes. Therefore the validity of (4.8) and (4.9) is much larger than that specified by (4.13). However, if the oblique modes are 'exponentially' small, Wundrow *et al.* (1994) show that the planar mode can become nonlinear (described by the strongly nonlinear critical layer equations, cf. Goldstein, Durbin & Leib 1987) before the oblique modes can produce a feedback effect on it. The oblique modes in this stage evolve over a faster time (or spatial) scale. Finally the disturbance enters a fully interactive stage with both the planar and oblique modes evolving over an inviscid time scale. While Wundrow *et al.* (1994) drew this conclusion for the resonant triad of long-wavelength Rayleigh modes, it is also true for the resonant triad of $O(1)$-wavelength Rayleigh modes in flows with *regular* critical layers. However, if the critical layer is *singular*, following the parametric resonance is the stage at which the development of the planar mode is governed by an evolution equation of Hickernell (1984) type. Because in this case the amplitude of the plane mode can either develop a finite singularity or equilibrate, the subsequent stage could be different from that of Wundrow *et al.* (1994). This issue may deserve further investigation.

5. Applications

Although the analysis in §§3,4 is performed for the Rayleigh instability waves evolving in time, the final amplitude equations also apply to Rayleigh waves which evolve spatially in the streamwise direction. Moreover, after dropping the linear terms the amplitude equations also describe the fully coupled stage of a resonant triad of TS waves identified by Goldstein (1994). In both cases, to obtain the coefficients only a minor modification to the outer analysis is needed. The jump across the critical layer can be borrowed directly from §3 if we define the slow streamwise variable carefully. In the following, we demonstrate this using a mixing layer and the Blasius boundary layer as examples.

5.1. *Mixing layer: spatial development of resonant triad of Rayleigh modes*

A mixing layer, which forms between two streams of different velocities $U^{(1)}$ and $U^{(2)}$, can support Rayleigh modes with wavelength comparable to its local thickness, say δ^*. We adopt the standard non-dimensionalization based on δ^* and a reference velocity $U_0 = (U^{(1)} - U^{(2)})/2$. The Reynolds number is $R = U_0\delta^*/\nu$. For the disturbance in form of a subharmonic resonant triad, it follows from the Squire transform that all three waves involved become neutral at the same streamwise location. The fully interactive resonance occurs at an $O(\epsilon^{1/3}R)$ distance upstream of the neutral position so that the local frequencies of the waves deviate from the corresponding neutral values by $O(\epsilon^{1/3})$, where ϵ is the magnitude of the oblique modes. Therefore in the main part of the shear layer, the vertical velocity of the disturbance can be written as

$$v = \epsilon A(x_1)\bar{v}_1 e^{i\alpha X}\cos\beta z + \epsilon^{4/3}[B(x_1)e^{2i\alpha X}\bar{\phi}_1 + \bar{v}_2 e^{i\alpha X}\cos\beta z + \ldots] \\ +\epsilon^{5/3}[\bar{\phi}_2 e^{2i\alpha X} + \ldots] + \text{c.c.} + \ldots , \tag{5.1}$$

where $\bar{v}_1 = \bar{\phi}_1$, and $X = x - (\bar{U}_c + \epsilon^{1/3}\hat{S})t$ with $\bar{U}_c = c$ being the phase speed of the neutral modes. The slow variable describing the streamwise development of the disturbance is defined by

$$x_1 = \epsilon^{1/3}\bar{U}_c^{-1}x , \tag{5.2}$$

where we have introduced a factor $\bar{U}_c^{-1}$ so that we can make use of the jump conditions obtained in §3 directly without going through a detailed analysis.

The eigenfunction $\bar{v}_1$ satisfies Rayleigh's equation. For the mixing layer, the critical level y_c coincides with the inflexion point so that $\bar{U}_c'' = 0$. As $\eta = y - y_c \to \pm 0$,

$$\bar{v}_1 \to 1 + \frac{1}{2}\left(\bar{\alpha}^2 + \frac{\bar{U}_c'''}{\bar{U}_c'}\right)\eta^2 + a\eta + \ldots ,$$

where $\bar{U}_c'$ and $\bar{U}_c'''$ represent the first and third derivatives (with respect to y) of the basic flow at the critical level. The constant $a = \bar{v}_1'(0)$, but its value is not needed in the following.

The functions $\bar{v}_2$ and $\bar{\phi}_2$ are governed by inhomogeneous Rayleigh equations, and as $\eta \to \pm 0$,

$$\bar{v}_2 \to d + \frac{\bar{U}_c'''}{\alpha \bar{U}_c' |\bar{U}_c'|} \left[\frac{\mathrm{d}A}{\mathrm{d}x_1} - \mathrm{i}\alpha \hat{S} A \right] \eta \log \eta + c^{\pm} y + \dots ,$$

$$\bar{\phi}_2 \to D + \frac{\bar{U}_c'''}{2\alpha \bar{U}_c' |\bar{U}_c'|} \left[\frac{\mathrm{d}B}{\mathrm{d}x_1} - 2\mathrm{i}\alpha \hat{S} B \right] \eta \log \eta + C^{\pm} y + \dots .$$

The solvability conditions for $\bar{v}_2$ and $\bar{\phi}_2$ lead to

$$2\mathrm{i}\alpha \bar{U}_c^{-1} J_1 \frac{\mathrm{d}A}{\mathrm{d}x_1} - \mathrm{i}\alpha^{-1} \left[\frac{\mathrm{d}A}{\mathrm{d}x_1} - \mathrm{i}\alpha \hat{S} A \right] J_2 = c^+ - c^- , \tag{5.3}$$

$$4\mathrm{i}\alpha \bar{U}_c^{-1} J_1 \frac{\mathrm{d}B}{\mathrm{d}x_1} - \mathrm{i}(2\alpha)^{-1} \left[\frac{\mathrm{d}B}{\mathrm{d}x_1} - 2\mathrm{i}\alpha \hat{S} B \right] J_2 = C^+ - C^- , \tag{5.4}$$

where

$$J_1 = \int_{-\infty}^{+\infty} \bar{v}_1^2 \mathrm{d}y , \quad J_2 = \int_{-\infty}^{+\infty} \frac{\bar{U}'' \bar{v}_1^2}{(\bar{U} - c)^2} \mathrm{d}y . \tag{5.5}$$

Within the critical layer, the solution takes the form (3.1)–(3.4). The critical layer operator is

$$\frac{\partial}{\partial x_1} + n\mathrm{i}\alpha (\bar{U}_c' Y - \hat{S}) - \lambda \frac{\partial^2}{\partial Y^2}$$

which is the same as (3.7) if we identify x_1 with t_1, and $\hat{S}$ with $\bar{U}_\tau \tau_1$. Moreover, the relevant nonlinear forcing terms that contribute to the jumps are the same as those in §3. Therefore, we conclude that the nonlinear parts in the jumps $(c^+ - c^-)$ and $(C^+ - C^-)$ are exactly the same as those in (3.74) and (3.114) with b_j being taken as unity, while the linear parts are replaced by

$$-\frac{\pi \bar{U}_c'''}{\alpha \bar{U}_c' |\bar{U}_c'|} \left[\frac{\mathrm{d}A}{\mathrm{d}x_1} - \mathrm{i}\alpha \hat{S} A \right] , \quad \text{and} \quad -\frac{\pi \bar{U}_c'''}{2\alpha \bar{U}_c' |\bar{U}_c'|} \left[\frac{\mathrm{d}B}{\mathrm{d}x_1} - 2\mathrm{i}\alpha \hat{S} B \right]$$

respectively; they correspond to the phase shift of π at the logarithmic branch point. Substituting the jumps so obtained into (5.3) and (5.4), we arrive at the same amplitude equations as (4.1) and (4.3) provided t_1 is interpreted as x_1, of course. The coefficients are now given by

$$\kappa_a = f_0 \hat{S} / f_1 , \quad g_{11} = 4\pi \mathrm{i} \alpha^2 \bar{U}_c' |\bar{U}_c'| \sin^2\theta / f_1 , \quad g_{12} = \pi \alpha^3 |\bar{U}_c'|^3 \sin^2\theta / f_1 , \tag{5.6}$$

$$\kappa_b = f_0 \hat{S} / f_2 , \quad g_{21} = \tfrac{1}{2} \pi \alpha^3 |\bar{U}_c'|^3 \sin^2\theta / f_2 , \quad g_{22} = 2\pi \mathrm{i} \alpha^4 \bar{U}_c'^3 |\bar{U}_c'| \sin^2\theta / f_2 , \tag{5.7}$$

where

$$f_0 = J_2 + \frac{\pi \mathrm{i} \bar{U}_c'''}{\bar{U}_c' |\bar{U}_c'|} , \quad f_1 = 2\mathrm{i}\alpha \bar{U}_c^{-1} J_1 - \mathrm{i}\alpha^{-1} f_0 , \quad f_2 = 4\mathrm{i}\alpha \bar{U}_c^{-1} J_1 - \mathrm{i}(2\alpha)^{-1} f_0 .$$

Note that the expressions for the coefficients are valid for shear flows with any profile. In general, they must be evaluated numerically by solving the (homogeneous) Rayleigh equation. However, for the shear flow with the profile $\bar{U} = \tanh y$, we have

$$\bar{U}_c = \frac{U^{(1)} + U^{(2)}}{U^{(1)} - U^{(2)}} , \quad \bar{U}_c' = 1 , \quad \bar{U}_c''' = -2 , \quad \alpha = \tfrac{1}{2} , \quad J_1 = 2 , \quad J_2 = 0 . \tag{5.8}$$

In passing, we note that the temporal development of a resonant triad on such a profile was considered by Mallier & Maslowe (1994) in the inviscid limit. Substitution of (5.8) into (5.6) and (5.7) gives

$$\kappa_a = \tfrac{1}{2}\mathrm{i}(1-\tfrac{1}{2}\mathrm{i}\chi)^{-1}\hat{S}\ , \quad g_{11} = -\tfrac{3}{16}\mathrm{i}(1-\tfrac{1}{2}\mathrm{i}\chi)^{-1}\ , \quad g_{12} = -\tfrac{3}{128}(1-\tfrac{1}{2}\mathrm{i}\chi)^{-1}\ , \tag{5.9}$$

$$\kappa_b = \mathrm{i}(1-2\mathrm{i}\chi)^{-1}\hat{S}\ , \quad g_{21} = -\tfrac{3}{128}(1-2\mathrm{i}\chi)^{-1}\ , \quad g_{22} = -\tfrac{3}{64}\mathrm{i}(1-2\mathrm{i}\chi)^{-1}\ , \tag{5.10}$$

where $\chi = 1/\pi\bar{U}_c$.

5.2. *Resonant triad of TS waves in the Blasius boundary layer*

A resonant triad consisting of TS waves in the upper-branch scaling regime was studied by Mankbadi *et al.* (1993). The dominant interactions are found to take place in the critical layer. But in contrast to the resonant triad of Rayleigh waves, the critical layer is of equilibrium type, although an unsteady diffusion layer has to be introduced to accommodate the cubic interaction between the oblique waves (Wu 1993). Recently, Goldstein (1994) observed that if the magnitude of the oblique waves is exponentially small at the start of the parametric resonance, then following the parametric resonance the disturbance can evolve into a regime in which the critical layer becomes of non-equilibrium type and the interactions become fully coupled. He also shows that in this regime, the development of the disturbance is governed by the amplitude equations of the form (4.1) and (4.3) although the complete equations are not derived. In this subsection, we show that such amplitude equations and the associated coefficients can be readily obtained by a minor modification of the analysis in §2. In the following, the non-dimensionalization is based on the thickness of the boundary layer and the free-stream velocity.

In the parametric resonance stage, the planar and the oblique modes all evolve on the slow streamwise variable (Mankbadi *et al.* 1993)

$$x_1 = \sigma^4 x\ , \tag{5.11}$$

where the small parameter $\sigma = (\omega^* \nu/U_0^2)^{1/6}$ with ω^* being the dimensional frequency of the plane mode. The (local) Reynolds number R is scaled as $R = \sigma^{-10}\bar{R}$. In particular, as a result of parametric resonance, the oblique waves experience a super-exponential growth while the planar wave still evolves exponentially, say with the growth rate $\kappa_b = \kappa_{br} + \mathrm{i}\kappa_{bi}$.

Goldstein (1994) observes that the critical layer dynamics of the oblique waves becomes non-equilibrium in nature when

$$x_1 = O(\kappa_{br}^{-1}\log\sigma^{-1})\ . \tag{5.12}$$

He shows that when x_1 is specified as above, but $(x_1 - \kappa_{br}^{-1}\log\sigma^{-1}) \gg \sigma$, there exists an intermediate regime in which the amplitude of the oblique waves takes the WKBJ form. Moreover, he shows that if the oblique waves have an appropriate magnitude at the start of the parametric resonance, they can produce a feedback effect on the planar mode when $(x_1 - \kappa_{br}^{-1}\log\sigma^{-1}) = O(\sigma)$. In this final stage, all the modes evolve on the faster spatial scale

$$\hat{x}_1 = \bar{c}^{-1}(x_1 - \kappa_{br}^{-1}\log\sigma^{-1})/\sigma\ , \tag{5.13}$$

where $\bar{c}$ is the (scaled) phase velocity of the disturbance. The flow is then described by a structure consisting of four layers: the main layer, the potential layer, the Tollmien layer and the critical layer, which now is non-equilibrium as well as viscous. The viscous Stokes layer adjacent to the wall now can be ignored because its contribution is a higher-order effect. In the main part of the boundary layer, the vertical velocity of the disturbance is

$$v = \delta\sigma A(\hat{x}_1)\bar{v}_1 e^{i\alpha X}\cos\beta Z + \epsilon\sigma B(\hat{x}_1)e^{2i\alpha X}\bar{\phi}_1 + \text{c.c.} + \dots , \tag{5.14}$$

where $\delta = \sigma^7$ and $\epsilon = \sigma^9$ as identified by Goldstein (1994). Here for convenience, we have defined

$$X = \sigma\alpha(x - \sigma\bar{c}t) , \quad Z = \sigma\beta z .$$

Other components can be expanded in a similar way. The solution in other layers takes a different form, but outside the critical layer the flow is linear up to the order of our interest, and the procedure for seeking the solution is basically the same as that in Bodonyi & Smith (1981). Matching the solution in the main layer with that in the potential and the Tollmien layers leads to

$$i(\cos\theta + \sec\theta)\bar{c}^{-1}\frac{dA}{d\hat{x}_1} = \bar{c}(c^+ - c^-) , \tag{5.15}$$

$$2i\bar{c}^{-1}\frac{dB}{d\hat{x}_1} = \bar{c}(C^+ - C^-) . \tag{5.16}$$

Since nonlinear interactions within the critical layer are exactly the same as described in §3, the jumps $(c^+ - c^-)$ and $(C^+ - C^-)$ are given by the nonlinear parts of (3.74) and (3.114) (with $b_j = \bar{c}$) respectively; the linear parts do not arise because the small curvature of the basic-flow profile at the critical level can produce jumps only at higher orders. Substituting such $(c^+ - c^-)$ and $(C^+ - C^-)$ into (5.15) and (5.16), we obtain the amplitude equations of the form (4.1) and (4.3) but without the linear terms. The associated coefficients are

$$g_{11} = \tfrac{6}{5}\pi\alpha^2\hat{\lambda}^2\bar{c}^4 , \quad g_{12} = -\tfrac{3}{10}\pi i\alpha^3\hat{\lambda}^3\bar{c}^5 , \tag{5.17}$$

$$g_{21} = -\tfrac{3}{16}\pi i\alpha^3\hat{\lambda}^3\bar{c}^5 , \quad g_{22} = \tfrac{3}{4}\pi\alpha^4\hat{\lambda}^4\bar{c}^6 , \tag{5.18}$$

where $\hat{\lambda}$ is the shear of the basic flow at the wall. The parameter s in the viscous kernels is now defined by

$$s = \tfrac{1}{3}\alpha^2\hat{\lambda}^2\bar{R}^{-1} . \tag{5.19}$$

Goldstein (1994) shows that the appropriate initial conditions follow from the asymptotic behaviour of the solution in the WKBJ stage, namely, as $\hat{x}_1 \to -\infty$,

$$A(\hat{x}_1) \to \hat{a}_0 e^{\hat{b}_0\hat{x}_1} , \quad B(\hat{x}_1) \to 1 , \tag{5.20}$$

where $\hat{a}_0$ is a real parameter representing the initial amplitude of the oblique modes, and $\hat{b}_0$ is a real constant determined by

$$\hat{b}_0 = g_{11}\int_0^{+\infty} \xi^2 e^{-2s\xi^3 - 2\hat{b}_0\xi} d\xi .$$

Therefore through the WKBJ and the parametric-resonance stages, the solution matches to the linear regime upstream; for details concerning the match between different stages, see Goldstein (1994).

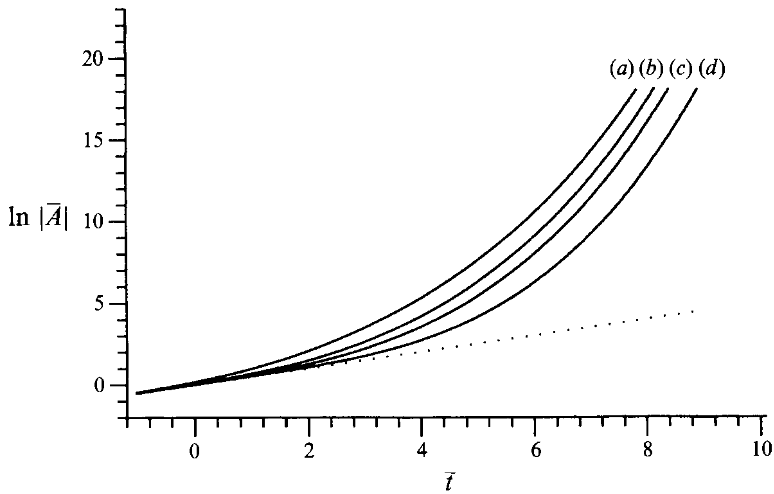

FIGURE 1. $\ln|\bar{A}|$ *vs.* $\bar{t}$ in the parametric-resonance regime ($\chi_0 = 0$): (*a*) $\lambda = 0$ (inviscid limit); (*b*) $\lambda = 5$; (*c*) $\lambda = 20$; (*d*) $\lambda = 120$. The parameters are $\alpha = 0.4$ and $\varphi_0 = 0$.

6. Study of the amplitude equations

6.1. *Finite-time singularity structure*

In the inviscid limit, solutions of (4.8) and (4.9) develop a finite-time singularity of the form (Goldstein & Lee 1992; Wu 1992)

$$\bar{A} \to \frac{a_0}{(t_s - \bar{t})^{3+i\sigma}}, \quad \bar{B} \to \frac{b_0}{(t_s - \bar{t})^{4+2i\sigma}} \quad \text{as} \quad \bar{t} \to t_s , \tag{6.1}$$

where a_0, b_0 are complex numbers and σ is a real number. The parameters σ, a_0 and t_s can be determined as described in Wu (1992). Although the above singularity is identified for the inviscid case, substitution of (6.1) into (4.8) and (4.9) shows that the structure is unaltered by viscous effects. The time at which the singularity occurs is delayed by viscosity, as our numerical results will show.

6.2. *Numerical study of the amplitude equations*

We integrate the (rescaled) amplitude equations (4.8) and (4.9) numerically. The finite-difference scheme that we use is the Adams–Moulton (implicit) method with sixth-order accuracy. While the kernels $K_a(\xi, \eta)$, $K_b(\xi, \eta, \zeta)$, etc. are rather complicated, they can be readily evaluated numerically using the Trapezoidal rule. Since in the viscous case these kernels decay exponentially as ξ, η and ζ tend to infinity, they are assigned a zero value when the arguments become sufficiently large. This can speed up the computation but without affecting the accuracy.

As in Wu *et al.* (1993), the integrals over the infinite domains (see (4.8), (4.9)) are approximated by those over large but finite domains. The sizes of the domains are determined by trial and error.

We first study the special case, i.e. the parametric resonance governed by (4.14) and (4.15). The coefficients that we use are those calculated for the oscillatory Stokes layer for $\alpha = 0.4$. The results are shown in figure 1. It is seen that the amplitude of the oblique modes quickly increases due to the parametric resonance. The viscosity has a stabilizing effect in that it acts to inhibit the growth. However, it does not alter the overall trend. The massive amplification that the oblique modes experience during

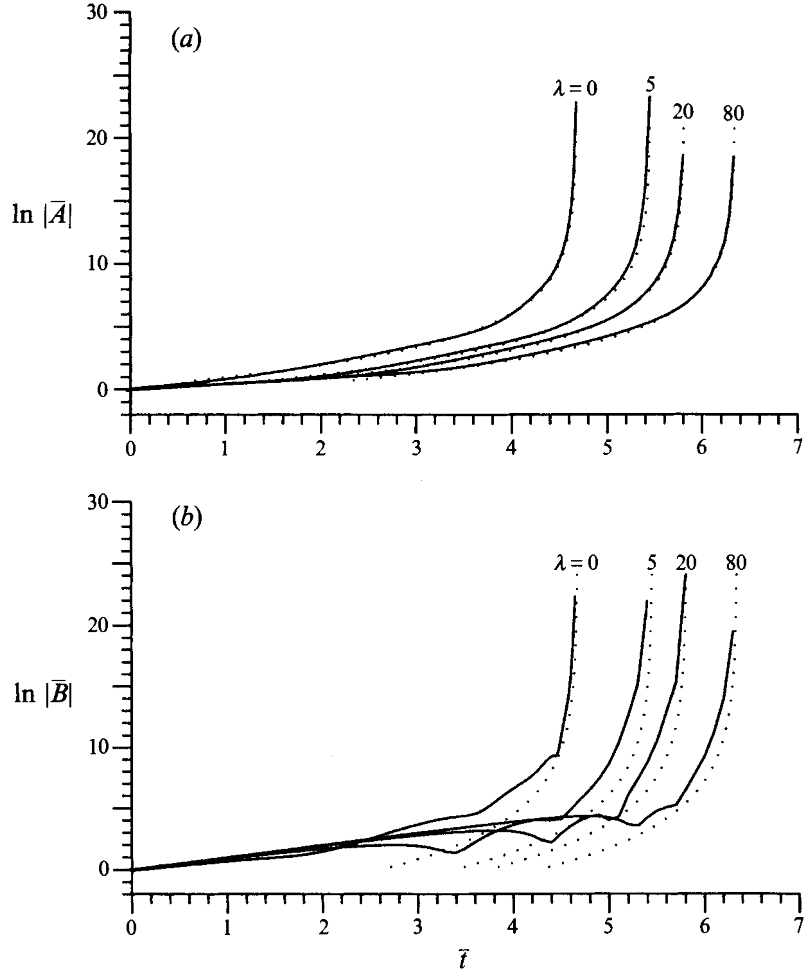

FIGURE 2. (a) $\ln|\bar{A}|$ and (b) $\ln|\bar{B}|$ vs. $\bar{t}$ for $\alpha = 0.6$ and $\lambda = 0, 5, 20, 80$. Solid lines: numerical solutions; dotted lines: local asymptotic solutions (6.1).

the parametric-resonance stage can soon lead to the fully coupled stage even though their initial magnitude is small.

The fully coupled amplitude equations (4.8) and (4.9) are then solved. In all the calculations presented below, we chose $\varphi_0 = 0$ and $\chi_0 = 0.1$. This size of χ_0 is chosen so that the development of the amplitudes can clearly reveal the linear, the parametric-resonance and the fully interactive stages. If χ_0 is too large, the parametric-resonance stage may be bypassed because the fully-interactive stage follows directly the linear stage (Wu 1992). On the other hand if χ_0 is too small, then we have to march the equations forward for a considerable time before entering the final stage and the computation hence becomes excessive. Of course if a quantitative comparison is to be performed, then χ_0, which represents the relative magnitude of the oblique waves to that of the planar mode, must be determined by experimental condition.

The result shown in figures 2(a) and 2(b) is for $\alpha = 0.6$. As illustrated, the solutions develop a singularity within a finite time. The viscosity effect is to delay the time at which the singularity occurs. We note that for $\alpha = 0.6$, $\sum g_{12} s^{-4/3} > 0$. For this case, if

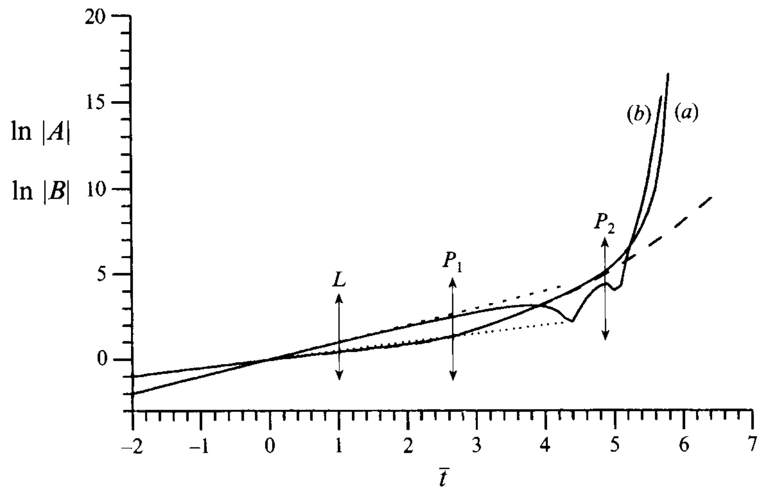

FIGURE 3. Stages that the disturbance evolves through: (*a*) $\ln|\bar{A}|$ *vs.* $\bar{t}$; (*b*) $\ln|\bar{B}|$ *vs.* $\bar{t}$. The dotted lines represent the linear solutions, and the dashed line represents the parametric-resonance solution. The parameters are $\alpha = 0.6$, $\lambda = 20$.

the amplitude A is decoupled from the amplitude B by removing the quadratic term, i.e. if the interaction involves only a pair of oblique waves, Wu *et al.* (1993) found that the singularity in A always occurs no matter how large the (scaled) viscosity is. It seems reasonable to expect that this is still true with the quadratic term being included in the present situation of resonant-triad interaction.

In order to follow the evolution of the disturbance closely, in figure 3, we plot the amplitudes of the planar and the oblique modes together. Four distinct regimes can be identified. Up to L is the linear stage where the modes evolve independently and all grow exponentially. The parametric-resonance stage starts from L. This is characterized by the fact that the quadratic resonance begins to affect the development of the oblique modes. The planar mode still follows linear theory up to P_1. While the parametric resonance ultimately enhances the growth of the oblique modes, initially it can cause the oblique modes to decay or to evolve at a rate smaller than the linear growth rate, depending on the parameters φ_0 and g_{11}. Starting from P_1, the oblique modes attain a sufficiently large amplitude to produce a feedback effect on the planar mode, causing the latter to deviate from the exponential growth. However up to P_2, such a deviation is not felt by the oblique modes, which continue to evolve as if the plane mode were growing exponentially. This behaviour is understandable because the growth rate of the oblique modes depends on the whole history rather than on the instantaneous amplitude so that the deviation of the planar mode from the linear theory in such a finite time (between P_1 and P_2) cannot outweigh the accumulated history effect. To indicate its main feature, it seems appropriate to refer the regime between P_1 and P_2 as the *extended parametric-resonance stage.* It is interesting to note that it is in this stage rather than in the pure parametric-resonance stage that the oblique modes are substantially amplified by quadratic resonance. We also note that between P_1 and P_2, the feedback effect on the planar mode is mainly produced by the cubic terms while the contribution from the quartic term is negligible, as in the inviscid case studied by Goldstein & Lee (1992). Starting from P_2, the self-interactions of the oblique modes come into play to induce a finite-time singularity, which is transmitted

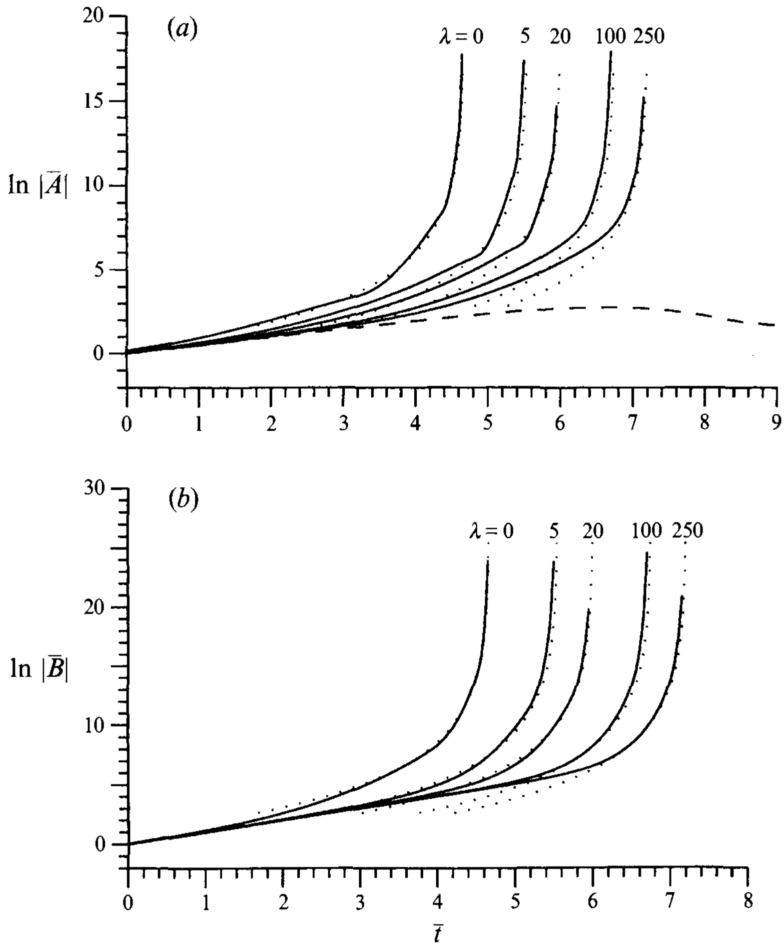

FIGURE 4. (*a*) $\ln|\bar{A}|$ and (*b*) $\ln|\bar{B}|$ *vs.* $\bar{t}$ for $\alpha = 0.4$ and $\lambda = 0, 5, 20, 100, 250$. Solid lines: numerical solutions; dotted lines: local asymptotic solutions (6.1). The dashed line represents the solution when the quadratic term is excluded from (4.8).

back to the planar mode through the feedback terms, leading to the formation of singularity in the amplitude B.

The next case that we examined is for $\alpha = 0.4$. Figures 4(*a*) and 4(*b*) show that the finite-time singularity again occurs. For this wavenumber, $\sum g_{12}s^{-4/3} < 0$; so if the quadratic term is removed from (4.8), the amplitude A decays exponentially when the viscosity is sufficiently large (Wu *et al.* 1993) as indicated by the dashed line in figure 4(*a*) which corresponds to $\lambda = 100$. However, with the presence of the quadratic term, the solution terminates at a finite-time singularity at the same size of λ. Further increasing the size of λ, we find that the singularity persists. It appears that for the fully coupled resonant–triad interaction, viscosity cannot eliminate the singularity because of the participation of the quadratic interaction. The viscous effect is to delay the time at which the singularity occurs.

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Appendix A

$$\hat{K}_v(\xi,\eta,\zeta) = \tfrac{1}{4}\left[2\xi^3 I^{(0)}(\xi,\eta) + \xi I_u^{(0,2)}(\xi,\eta) + \xi^2 I_v^{(0,2)}(\xi,\eta)\right]\mathrm{e}^{-s\xi^3-s(\zeta-\xi)^3}$$
$$+ \tfrac{1}{2}\sin^2\theta\eta(\xi+\eta)(2\xi+\eta)\mathrm{e}^{-s\xi^3-s\eta^3-s(\xi+\eta)^3-s(\zeta-\xi)^3}$$
$$-\sin^2\theta\int_0^\zeta\Big\{\Big[(\eta+\gamma)I_v^{(0,2)}(\xi+\eta,\gamma) + \Big(1-3s(\eta+\gamma)^3\Big)I_u^{(0,2)}(\xi+\eta,\gamma)\Big]\mathrm{e}^{-s(\eta+\gamma)^3}$$
$$+\Big[\gamma I_v^{(0,2)}(\xi,\eta+\gamma) + \Big(1-3s\gamma^3\Big)I_u^{(0,2)}(\xi,\eta+\gamma)\Big]\mathrm{e}^{-s\gamma^3}\Big\}\mathrm{e}^{-s(\xi-\gamma)^3-s(\zeta-\xi)^3}\mathrm{d}\gamma$$
$$+4\sin^4\theta\int_0^\zeta(\xi+\eta+\gamma)^2\mathrm{e}^{-s\gamma^3-s(\eta+\gamma)^3-s(\xi+\eta+\gamma)^3-s(\xi-\gamma)^3-s(\zeta-\xi)^3}\mathrm{d}\gamma$$
$$-6s\sin^2\theta\int_0^\zeta(\xi+\eta+\gamma)^3 I_v^{(2,0)}(\eta,\gamma)\mathrm{e}^{-s(\xi+\eta+\gamma)^3-s(\xi-\gamma)^3-s(\zeta-\xi)^3}\mathrm{d}\gamma\ . \tag{A 1}$$

$$\tilde{K}_v(\xi,\eta,\zeta) = -\tfrac{1}{2}\left[3(\xi+2\eta)I_v^{(2,0)}(\xi,\eta) + \xi(\xi+2\eta)^2 I^{(1)}(\xi,\eta)\right]\mathrm{e}^{s(\xi+2\eta)^3/3-s(\xi+2\eta+3\zeta)^3/3}$$
$$+\sin^2\theta\eta(\xi+\eta)(\xi+2\eta)\mathrm{e}^{-s(\xi+\eta)^3-s\eta^3+s(\xi+2\eta)^3/3-s(\xi+2\eta+3\zeta)^3/3}$$
$$-4\sin^2\theta\int_0^\zeta\Big\{\Big[\eta(\xi+2\eta+3\gamma)I^{(1)}(\eta,\gamma) + \Big(1-\tfrac{3}{2}s(\xi+\eta+\gamma)^3\Big)I_v^{(2,0)}(\eta,\gamma)\Big]\mathrm{e}^{-s(\xi+\eta+\gamma)^3}$$
$$+\Big[(\xi+\eta)(\xi+2\eta+3\gamma)I^{(1)}(\xi+\eta,\gamma) + \Big(1-\tfrac{3}{2}s(\eta+\gamma)^3\Big)I_v^{(2,0)}(\xi+\eta,\gamma)\Big]\mathrm{e}^{-s(\eta+\gamma)^3}$$
$$+\Big[\xi(\xi+2\eta+3\gamma)I^{(1)}(\xi,\eta+\gamma)+\Big(1-\tfrac{3}{2}s\gamma^3\Big)I_v^{(2,0)}(\xi,\eta+\gamma)\Big]\mathrm{e}^{-s\gamma^3}\Big\}\mathrm{e}^{s(\xi+2\eta+3\gamma)^3/3-s(\xi+2\eta+3\zeta)^3/3}\mathrm{d}\gamma$$
$$+4\sin^4\theta\int_0^\zeta\left[\gamma^2+(\eta+\gamma)^2+(\xi+\eta+\gamma)^2\right]\mathrm{e}^{-s(\xi+\eta+\gamma)^3-s(\eta+\gamma)^3-s\gamma^3+s(\xi+2\eta+3\gamma)^3/3-s(\xi+2\eta+3\zeta)^3/3}\mathrm{d}\gamma\ . \tag{A 2}$$

$$\hat{K}_w(\xi,\eta,\zeta) = -\tfrac{1}{4}\xi I_v^{(0,2)}(\xi,\eta)\mathrm{e}^{-s\xi^3-s(\zeta-\xi)^3} - \tfrac{3}{4}\sin^2\theta\eta(\xi+\eta)\mathrm{e}^{-s\xi^3-s\eta^3-s(\xi+\eta)^3-s(\zeta-\xi)^3}$$
$$-\tfrac{1}{4}\sin^2\theta\textstyle\int_0^\zeta\Big\{\Big[2(\xi+\eta)I^{(0)}(\xi+\eta,\gamma)-I_v^{(0,2)}(\xi+\eta,\gamma)-6s(\eta+\gamma)^2 I_u^{(0,2)}(\xi+\eta,\gamma)\Big]\mathrm{e}^{-s(\eta+\gamma)^3}$$
$$+\Big[2\xi I^{(0)}(\xi,\eta+\gamma)-I_v^{(0,2)}(\xi,\eta+\gamma)-6s\gamma^2 I_u^{(0,2)}(\xi,\eta+\gamma)\Big]\mathrm{e}^{-s\gamma^3}\Big\}\mathrm{e}^{-s(\xi-\gamma)^3-s(\zeta-\xi)^3}\mathrm{d}\gamma$$
$$-2\sin^4\theta\int_0^\zeta(\xi+\eta+\gamma)\mathrm{e}^{-s(\xi+\eta+\gamma)^3-s(\eta+\gamma)^3-s\gamma^3-s(\xi-\gamma)^3-s(\zeta-\xi)^3}\mathrm{d}\gamma$$
$$+3s\sin^2\theta\int_0^\zeta(\xi+\eta+\gamma)^2 I_v^{(2,0)}(\eta,\gamma)\mathrm{e}^{-s(\xi+\eta+\gamma)^3-s(\xi-\gamma)^3-s(\zeta-\xi)^3}\mathrm{d}\gamma\ . \tag{A 3}$$

$$\tilde{K}_w(\xi,\eta,\zeta) = \tfrac{1}{4}\xi(\xi+2\eta)I^{(1)}(\xi,\eta)\mathrm{e}^{s(\xi+2\eta)^3/3-s(\xi+2\eta+3\zeta)^3/3}$$
$$-\sin^2\theta\eta(\xi+\eta)\mathrm{e}^{-s(\xi+\eta)^3-s\eta^3+s(\xi+2\eta)^3/3-s(\xi+2\eta+3\zeta)^3/3}$$
$$+\tfrac{1}{2}\sin^2\theta\int_0^\zeta\Big\{\Big[\eta I^{(1)}(\eta,\gamma)-12s(\xi+\eta+\gamma)^2I_v^{(2,0)}(\eta,\gamma)\Big]\mathrm{e}^{-s(\xi+\eta+\gamma)^3}$$
$$+\Big[(\xi+\eta)I^{(1)}(\xi+\eta,\gamma)-12s(\eta+\gamma)^2I_v^{(2,0)}(\xi+\eta,\gamma)\Big]\mathrm{e}^{-s(\eta+\gamma)^3}$$
$$+\Big[\xi I^{(1)}(\xi,\eta+\gamma)-12s\gamma^2I_v^{(2,0)}(\xi,\eta+\gamma)\Big]\mathrm{e}^{-s\gamma^3}\Big\}\mathrm{e}^{s(\xi+2\eta+3\gamma)^3/3-s(\xi+2\eta+3\zeta)^3/3}\mathrm{d}\gamma$$
$$-2\sin^4\theta\int_0^\zeta(\xi+2\eta+3\gamma)\mathrm{e}^{-s\gamma^3-s(\eta+\gamma)^3-s(\xi+\eta+\gamma)^3+s(\xi+2\eta+3\gamma)^3/3-s(\xi+2\eta+3\zeta)^3/3}\mathrm{d}\gamma\,. \tag{A 4}$$

In the inviscid limit ($\lambda = 0$), the functions $\hat{K}_v(\xi,\eta,\zeta)$, etc. simplify to

$$\hat{K}_v(\xi,\eta,\zeta) = \tfrac{1}{2}\xi^2(2\xi+\eta)+\tfrac{1}{2}\sin^2\theta[-(2\xi+\eta)(2\xi+4\eta+3\zeta)\zeta+\eta(\xi+\eta)(4\xi+\eta)]$$
$$+4\sin^4\theta[-\zeta^3+(\xi-\eta)\zeta^2+\xi(\xi+2\eta)\zeta]\,, \tag{A 5}$$

$$\tilde{K}_v(\xi,\eta,\zeta) = -\xi(2\xi+\eta)(\xi+2\eta)$$
$$-\sin^2\theta[12(\xi+\eta)\zeta^2+4(4\xi^2+8\xi\eta+6\eta^2)\zeta+5\eta(\xi+\eta)(\xi+2\eta)]$$
$$+4\sin^4\theta[-3\zeta^3-3(\xi+2\eta)\zeta^2+(\xi^2-2\xi\eta-2\eta^2)\zeta]\,, \tag{A 6}$$

$$\hat{K}_w(\xi,\eta,\zeta) = -\tfrac{1}{4}\xi^2-\tfrac{1}{4}\sin^2\theta[(2\xi+\eta)\zeta+(7\xi+\eta)\eta]-\sin^4\theta(2\xi+\eta)\zeta\,, \tag{A 7}$$

$$\tilde{K}_w(\xi,\eta,\zeta) = \tfrac{1}{4}\xi(\xi+2\eta)+\sin^2\theta(\xi+\eta)(\zeta-\xi)-\sin^4\theta[3\zeta^2+2(\xi+2\eta)\zeta]\,. \tag{A 8}$$

Appendix B

$$K_a(\xi,\eta) = \tilde{K}^{(0)}(\xi,\eta)(2\xi^3+\xi^2\eta)$$
$$+2\sin^2\theta\left\{\tilde{K}^{(0)}(\xi,\eta)\int_0^\eta[\xi^2+2\xi(\eta-\zeta)]\mathrm{e}^{-2s\zeta^3-3s\xi\zeta^2}\mathrm{d}\zeta\right.$$
$$+\tilde{K}^{(0)}(\xi,\eta)\int_0^\xi[\zeta(2\eta+3\zeta)-\xi(\xi+2\eta+2\zeta)]\mathrm{e}^{-3s\xi\zeta^2}\mathrm{d}\zeta$$
$$+2\tilde{K}^{(1)}(\xi,\eta)\int_0^\xi\eta\zeta[1+6s(\xi-\zeta)(\xi+\eta+\zeta)^2]\Pi_0(\xi,\eta,\zeta)\mathrm{d}\zeta$$
$$\left.+\tilde{K}^{(1)}(\xi,\eta)\int_0^\xi[(\eta+\zeta)(\eta+3\zeta)-(\xi+\eta)(\xi+\eta+2\zeta)]\mathrm{e}^{-3s(\xi+\eta)(2\eta+\zeta)\zeta}\mathrm{d}\zeta\right\}$$
$$+8\sin^4\theta\left\{\tilde{K}^{(0)}(\xi,\eta)\int_0^\xi\mathrm{d}\zeta\mathrm{e}^{-3s\xi\zeta^2}\int_0^{\eta+\zeta}(\upsilon-\eta-\zeta)[1+6s(\xi-\zeta)\zeta^2]\mathrm{e}^{-s(2\upsilon^3+3\xi\upsilon^2)}\mathrm{d}\upsilon\right.$$
$$+2\tilde{K}^{(1)}(\xi,\eta)\int_0^\xi\mathrm{d}\zeta\Pi_0(\xi,\eta,\zeta)\int_0^\zeta(\zeta-\upsilon)[1+6s(\xi-\zeta)(\xi+\eta+\zeta)^2]\mathrm{e}^{s(2\upsilon^3+3\eta\upsilon^2)}\mathrm{d}\upsilon$$

$$+\tilde{K}^{(1)}(\xi,\eta)\int_0^{\xi} \mathrm{d}\zeta \mathrm{e}^{-3s(\xi+\eta)(2\eta+\zeta)\zeta}\int_0^{\zeta}(\upsilon-\zeta)[1+6s(\xi-\zeta)(\eta+\zeta)^2]\mathrm{e}^{-s[2\upsilon^3+3(\xi+\eta)\upsilon^2]}\mathrm{d}\upsilon\Big\} , \tag{B 1}$$

where

$$\tilde{K}^{(0)}(\xi,\eta)=\mathrm{e}^{-s(2\xi^3+3\xi^2\eta)} , \quad \tilde{K}^{(1)}(\xi,\eta)=\mathrm{e}^{-s[\xi^3+\eta^3+(\xi+\eta)^3]} ,$$
$$\Pi_0(\xi,\eta,\zeta)=\mathrm{e}^{-s(4\zeta^3+6\xi\zeta^2+9\eta\zeta^2+6\xi\eta\zeta+6\eta^2\zeta)} .$$

$$\begin{aligned}K_{21}(\xi,\eta)=&16\xi(\xi+\eta)(2\xi+\eta)\mathrm{e}^{-2s(2\xi+\eta)^3+4s\xi^3}\\&+4\sin^2\theta\xi(2\xi+\eta)(3\xi+2\eta)\mathrm{e}^{-2s(2\xi+\eta)^3}\\&-8\sin^2\theta\int_0^{\xi}(2\xi+\eta)\left[(2\xi+3\eta+3\gamma)-3s\gamma^3(2\xi+2\eta+\gamma)\right]\\&\times\mathrm{e}^{-s\gamma^3-2s(2\xi+\eta)^3+s(2\xi-\gamma)^3-4s(\xi-\gamma)^3}\mathrm{d}\gamma .\end{aligned} \tag{B 2}$$

$$\begin{aligned}K_{22}(\xi,\eta)=&32\xi^3\mathrm{e}^{-12s\xi^3-12s\xi^2\eta}+4\sin^2\theta(\xi+\eta)(2\xi+\eta)(3\xi+\eta)\mathrm{e}^{-2s(2\xi+\eta)^3}\\&-4\sin^2\theta\eta(\xi+\eta)(3\xi+\eta)\mathrm{e}^{-2s\eta^3/3-4s(3\xi+\eta)^3/3}\\&+8\sin^2\theta\xi\eta(2\xi+\eta)\mathrm{e}^{-s\eta^3-s(2\xi+\eta)^3-4s\xi^3}\\&-8\sin^2\theta\int_0^{\xi}(2\xi+\eta)\left[(2\xi+2\eta+3\gamma)-3s(\eta+\gamma)^3(2\xi+\eta+\gamma)\right]\\&\times\mathrm{e}^{-s(\eta+\gamma)^3-2s(2\xi+\eta)^3+s(2\xi+\eta-\gamma)^3-4s(\xi-\gamma)^3}\mathrm{d}\gamma\\&+8\sin^2\theta\int_0^{\xi}\eta\left[(2\xi-\gamma)-3s(\eta+\gamma)(2\xi+\eta+\gamma)^3\right]\\&\times\mathrm{e}^{-s(2\xi+\eta+\gamma)^3-2s\eta^3/3-s(\eta+3\gamma)^3/3-4s(\xi-\gamma)^3}\mathrm{d}\gamma .\end{aligned} \tag{B 3}$$

Appendix D

$$\begin{aligned}K_b(\xi,\eta,\zeta)=&-2\left[\zeta\hat{K}_v(\eta+2\zeta,\xi,\eta)+2\zeta^2\hat{K}_w(\eta+2\zeta,\xi,\eta)\right]\mathrm{e}^{-4s\zeta^3}\\&+\tfrac{1}{3}\sin^2\theta(\xi+2\eta+3\zeta)\left[\tilde{K}_v(\xi,\eta,\zeta)-(\xi+2\eta+3\zeta)\tilde{K}_w(\xi,\eta,\zeta)\right]\mathrm{e}^{-s(\xi+2\eta+3\zeta)^3}\\&-\sin^2\theta\Big\{\zeta\left[\hat{K}_v(\eta+2\zeta,\xi,\eta+\zeta)+\zeta\hat{K}_w(\eta+2\zeta,\xi,\eta+\zeta)\right]\mathrm{e}^{-s\zeta^3}\\&\quad+(\eta+\zeta)\left[\hat{K}_v(\eta+2\zeta,\xi+\eta,\zeta)+(\eta+\zeta)\hat{K}_w(\eta+2\zeta,\xi+\eta,\zeta)\right]\mathrm{e}^{-s(\eta+\zeta)^3}\\&\quad+(\xi+\eta+\zeta)\left[\hat{K}_v(\xi+\eta+2\zeta,\eta,\zeta)+(\xi+\eta+\zeta)\hat{K}_w(\xi+\eta+2\zeta,\eta,\zeta)\right]\mathrm{e}^{-s(\xi+\eta+\zeta)^3}\Big\}\\&+\tfrac{1}{3}\sin^2\theta\textstyle\int_0^{\zeta}\big\{\left[2(\xi+2\eta+6\zeta-3\gamma)-6s(\xi+2\eta+3\gamma)(\xi+2\eta+2\zeta+\gamma)^3\right]\tilde{K}_w(\xi,\eta,\gamma)\\&\quad+\left[1+6s(\xi+2\eta+2\zeta+\gamma)^3\right]\tilde{K}_v(\xi,\eta,\gamma)\big\}\,\mathrm{e}^{-s(\xi+2\eta+2\zeta+\gamma)^3-4s(\zeta-\gamma)^3}\mathrm{d}\gamma\\&+\sin^2\theta\int_0^{\zeta}\Big\{\Big\{[2(4\zeta-3\gamma)-6s\gamma^3(2\zeta-\gamma)]\hat{K}_w(\eta+2\zeta,\xi,\eta+\gamma)\\&\quad+(3-6s\gamma^3)\hat{K}_v(\eta+2\zeta,\xi,\eta+\gamma)\Big\}\mathrm{e}^{-s\gamma^3}\end{aligned}$$

$$+\Big\{[2(4\zeta-3\gamma+\eta)-6s(\eta+\gamma)^3(2\zeta-\gamma+\eta)]\hat{K}_w(\eta+2\zeta,\xi+\eta,\gamma)$$
$$+[3-6s(\eta+\gamma)^3]\hat{K}_v(\eta+2\zeta,\xi+\eta,\gamma)\Big\}e^{-s(\eta+\gamma)^3}$$
$$+\Big\{[2(4\zeta-3\gamma+\xi+\eta)-6s(\xi+\eta+\gamma)^3(2\zeta-\gamma+\xi+\eta)]\hat{K}_w(\xi+\eta+2\zeta,\eta,\gamma)$$
$$+[3-6s(\xi+\eta+\gamma)^3]\hat{K}_v(\xi+\eta+2\zeta,\eta,\gamma)\Big\}e^{-s(\xi+\eta+\gamma)^3}\Bigg\}e^{-4s(\zeta-\gamma)^3}\,d\gamma$$

$$+\sin^2\theta\int_0^\zeta\Big\{(\xi+2\eta+2\zeta)\big[1+3s(\xi+2\eta+2\zeta)^3\big]I^{(0)}(\xi+2\eta+2\zeta,\gamma)I_v^{(2,0)}(\xi,\eta+\gamma)$$
$$-\xi\big[1-\tfrac{3}{2}s(\xi+2\eta+2\gamma)^3\big]I_u^{(0,2)}(\xi+2\eta+2\zeta,\gamma)I^{(1)}(\xi,\eta+\gamma)$$
$$-\tfrac{1}{2}\xi(\xi+2\eta+4\gamma-2\zeta)I_v^{(0,2)}(\xi+2\eta+2\zeta,\gamma)I^{(1)}(\xi,\eta+\gamma)$$
$$+(\xi+\eta+2\zeta)\big[1+3s(\xi+\eta+2\zeta)^3\big]I^{(0)}(\xi+\eta+2\zeta,\eta+\gamma)I_v^{(2,0)}(\xi+\eta,\gamma)$$
$$-(\xi+\eta)\big[1-\tfrac{3}{2}s(\xi+\eta+2\gamma)^3\big]I_u^{(0,2)}(\xi+\eta+2\zeta,\eta+\gamma)I^{(1)}(\xi+\eta,\gamma)$$
$$-\tfrac{1}{2}(\xi+\eta)(\xi+\eta+4\gamma-2\zeta)I_v^{(0,2)}(\xi+\eta+2\zeta,\eta+\gamma)I^{(1)}(\xi+\eta,\gamma)$$
$$+(\eta+2\zeta)\big[1+3s(\eta+2\zeta)^3\big]I^{(0)}(\eta+2\zeta,\xi+\eta+\gamma)I_v^{(2,0)}(\eta,\gamma)$$
$$-\eta\big[1-\tfrac{3}{2}s(\eta+2\gamma)^3\big]I_u^{(0,2)}(\eta+2\zeta,\xi+\eta+\gamma)I^{(1)}(\eta,\gamma)$$
$$-\tfrac{1}{2}\eta(\eta+4\gamma-2\zeta)I_v^{(0,2)}(\eta+2\zeta,\xi+\eta+\gamma)I^{(1)}(\eta,\gamma)\Big\}e^{-4s(\zeta-\gamma)^3}\,d\gamma$$

$$+\tfrac{1}{2}\sin^2\theta\,\zeta(\xi+2\eta+2\zeta)\left\{\xi(\xi+2\eta)I^{(1)}(\xi,\eta)+2I_v^{(2,0)}(\xi,\eta)\right\}e^{-s(\xi+2\eta+2\zeta)^3-4s\zeta^3}$$
$$-\tfrac{1}{4}\sin^4\theta\,\eta(\xi+\eta)(\xi+2\eta+2\zeta)(\xi+2\eta+8\zeta)e^{-s(\xi+\eta)^3-s\eta^3-s(\xi+2\eta+2\zeta)^3-4s\zeta^3}$$
$$-\tfrac{1}{2}\sin^2\theta\Big\{(\xi+\eta)\zeta\big[I_u^{(0,2)}(\xi+\eta+2\zeta,\eta)+2(\xi+\eta+2\zeta)^2I^{(0)}(\xi+\eta+2\zeta,\eta)\big]e^{-s(\xi+\eta)^3}$$
$$+\eta\zeta\big[I_u^{(0,2)}(\eta+2\zeta,\xi+\eta)+2(\eta+2\zeta)^2I^{(0)}(\eta+2\zeta,\xi+\eta)\big]e^{-s\eta^3}\Big\}e^{-4s\zeta^3}$$

$$-2\sin^6\theta\int_0^\zeta(\xi+2\eta+2\zeta+\gamma)\Big[(\xi+\eta+\gamma)(\eta+2\zeta)+(\eta+\gamma)(\xi+\eta+2\zeta)+(\xi+2\eta+2\zeta)\gamma\Big]$$
$$e^{-s(\xi+2\eta+2\zeta+\gamma)^3-s(\eta+\gamma)^3-s\gamma^3-4s(\zeta-\gamma)^3-s(\xi+\eta+\gamma)^3}\,d\gamma$$

$$-\tfrac{1}{4}\sin^4\theta\int_0^\zeta\Bigg\{\Big\{2(\eta+2\zeta)\Big[4(\eta+2\zeta)^2+(\eta+\gamma)^2+\gamma^2\Big]I^{(0)}(\eta+2\zeta,\xi+\eta+\gamma)$$
$$-4\Big[(\eta+2\gamma)-3s\gamma(\eta+\gamma)\Big(\eta^2+3\gamma(\eta+\gamma)\Big)\Big]I_u^{(0,2)}(\eta+2\zeta,\xi+\eta+\gamma)$$
$$-\Big[(\eta+\gamma)^2+\gamma^2\Big]I_v^{(0,2)}(\eta+2\zeta,\xi+\eta+\gamma)\Big\}e^{-s(\eta+\gamma)^3-s\gamma^3}$$
$$+\Big\{2(\xi+\eta+2\zeta)\Big[4(\xi+\eta+2\zeta)^2+(\xi+\eta+\gamma)^2+\gamma^2\Big]I^{(0)}(\xi+\eta+2\zeta,\eta+\gamma)$$
$$-4\Big[(\xi+\eta+2\gamma)-3s\gamma(\xi+\eta+\gamma)\Big((\xi+\eta)^2+3\gamma(\xi+\eta+\gamma)\Big)\Big]I_u^{(0,2)}(\xi+\eta+2\zeta,\eta+\gamma)$$
$$-\Big[(\xi+\eta+\gamma)^2+\gamma^2\Big]I_v^{(0,2)}(\xi+\eta+2\zeta,\eta+\gamma)\Big\}e^{-s(\xi+\eta+\gamma)^3-s\gamma^3}$$
$$+\Big\{2(\xi+2\eta+2\zeta)\Big[4(\xi+2\eta+2\zeta)^2+(\xi+\eta+\gamma)^2+(\eta+\gamma)^2\Big]I^{(0)}(\xi+2\eta+2\zeta,\gamma)$$
$$-4\Big[(\xi+2\eta+2\gamma)-3s(\eta+\gamma)(\xi+\eta+\gamma)\Big(\xi^2+3(\eta+\gamma)(\xi+\eta+\gamma)\Big)\Big]I_u^{(0,2)}(\xi+2\eta+2\zeta,\gamma)$$

$$
- \Big[(\xi+\eta+\gamma)^2 + (\eta+\gamma)^2\Big] I_v^{(0,2)}(\xi+2\eta+2\zeta,\gamma)\Big\} e^{-s(\xi+\eta+\gamma)^3 - s(\eta+\gamma)^3} \Bigg\} e^{-4s(\zeta-\gamma)^3} \mathrm{d}\gamma
$$

$$
- 2\sin^4\theta \int_0^\zeta \Bigg\{ \Big\{ \Big[3s(\xi+2\eta+2\zeta+\gamma)\gamma\Big(\gamma(\xi+2\eta+2\zeta) - \tfrac{1}{3}(\xi+2\eta+2\zeta+\gamma)^2\Big)
$$

$$
+ (\xi+2\eta+2\zeta+\tfrac{4}{3}\gamma)\Big] I_v^{(2,0)}(\xi,\eta+\gamma) - \tfrac{3}{4}\xi(\xi+2\eta+2\zeta+\gamma)^2 I^{(1)}(\xi,\eta+\gamma)\Big\} e^{-s\gamma^3}
$$

$$
+ \Big\{ \Big[3s(\xi+2\eta+2\zeta+\gamma)(\eta+\gamma)\Big((\eta+\gamma)(\xi+\eta+2\zeta) - \tfrac{1}{3}(\xi+2\eta+2\zeta+\gamma)^2\Big)
$$

$$
+ (\xi+\tfrac{7}{3}\eta+2\zeta+\tfrac{4}{3}\gamma)\Big] I_v^{(2,0)}(\xi+\eta,\gamma)
$$

$$
- \tfrac{3}{4}(\xi+\eta)(\xi+2\eta+2\zeta+\gamma)^2 I^{(1)}(\xi+\eta,\gamma)\Big\} e^{-s(\eta+\gamma)^3}
$$

$$
+ \Big\{ \Big[3s(\xi+2\eta+2\zeta+\gamma)(\xi+\eta+\gamma)\Big((\xi+\eta+\gamma)(\eta+2\zeta) - \tfrac{1}{3}(\xi+2\eta+2\zeta+\gamma)^2\Big)
$$

$$
+ (\tfrac{4}{3}\xi+\tfrac{7}{3}\eta+2\zeta+\tfrac{4}{3}\gamma)\Big] I_v^{(2,0)}(\eta,\gamma) - \tfrac{3}{4}\eta(\xi+2\eta+2\zeta+\gamma)^2 I^{(1)}(\eta,\gamma)\Big\} e^{-s(\xi+\eta+\gamma)^3} \Bigg\}
$$

$$
e^{-s(\xi+2\eta+2\zeta+\gamma)^3 - 4s(\zeta-\gamma)^3} \mathrm{d}\gamma
$$

$$
+ \tfrac{1}{2}\sin^2\theta \Big[(\xi+2\eta+2\zeta)^2 I^{(0)}(\xi+2\eta+2\zeta,\zeta) I_v^{(2,0)}(\xi,\eta+\zeta)
$$

$$
+ (\xi+\eta+2\zeta)^2 I^{(0)}(\xi+\eta+2\zeta,\eta+\zeta) I_v^{(2,0)}(\xi+\eta,\zeta)
$$

$$
+ (\eta+2\zeta)^2 I^{(0)}(\eta+2\zeta,\xi+\eta+\zeta) I_v^{(2,0)}(\eta,\zeta)\Big]
$$

$$
+ \tfrac{1}{4}\sin^2\theta \Big[\xi(\xi+2\eta+2\zeta) I_u^{(0,2)}(\xi+2\eta+2\zeta,\zeta) I^{(1)}(\xi,\eta+\zeta)
$$

$$
+ (\xi+\eta)(\xi+\eta+2\zeta) I_u^{(0,2)}(\xi+\eta+2\zeta,\eta+\zeta) I^{(1)}(\xi+\eta,\zeta)
$$

$$
+ \eta(\eta+2\zeta) I_u^{(0,2)}(\eta+2\zeta,\xi+\eta+\zeta) I^{(1)}(\eta,\zeta)\Big]
$$

$$
+ \tfrac{4}{3}\sin^4\theta(\xi+2\eta+3\zeta)\Big[\zeta I_v^{(2,0)}(\xi,\eta+\zeta) e^{-s\zeta^3} + (\eta+\zeta) I_v^{(2,0)}(\xi+\eta,\zeta) e^{-s(\eta+\zeta)^3}
$$

$$
+ (\xi+\eta+\zeta) I_v^{(2,0)}(\eta,\zeta) e^{-s(\xi+\eta+\zeta)^3}\Big] e^{-s(\xi+2\eta+3\zeta)^3}
$$

$$
- \sin^4\theta \Big[(\eta+\zeta)(\xi+\eta+\zeta) I_u^{(0,2)}(\xi+2\eta+2\zeta,\zeta) e^{-s(\eta+\zeta)^3 - s(\xi+\eta+\zeta)^3}
$$

$$
+ \zeta(\xi+\eta+\zeta) I_u^{(0,2)}(\xi+\eta+2\zeta,\eta+\zeta) e^{-s\zeta^3 - s(\xi+\eta+\zeta)^3}
$$

$$
+ (\eta+\zeta)\zeta I_u^{(0,2)}(\eta+2\zeta,\xi+\eta+\zeta) e^{-s(\eta+\zeta)^3 - s\zeta^3}\Big]. \qquad \text{(D 1)}
$$

REFERENCES

BODONYI, R. J. & SMITH, F. T. 1981 The upper branch stability of the Blasius boundary layer, including non-parallel flow effects. *Proc. R. Soc. Lond.* A **375**, 65.

CRAIK, A. D. D. 1971 Non-linear resonant instability in boundary layers. *J. Fluid Mech.* **50**, 393.

CRAIK, A. D. D. 1985 *Wave Interactions and Fluid Flows.* Cambridge University Press.

GOLDSTEIN, M. E. 1994 Nonlinear interaction between oblique waves on nearly planar shear flows. *Phys. Fluids* A **6**, 42.

GOLDSTEIN, M. E. & CHOI, S.-W. 1989 Nonlinear evolution of interacting oblique waves on two-dimensional shear layers. *J. Fluid Mech.* **207**, 97. Corrigendum, *J. Fluid Mech.* **216**, 1990, 659.

GOLDSTEIN, M. E., DURBIN, P. A. & LEIB, S. J. 1987 Roll-up of vorticity in adverse-pressure-gradient boundary layers. *J. Fluid Mech.* **183**, 325.

GOLDSTEIN, M. E. & HULTGREN, L. S. 1989 Nonlinear spatial evolution of an externally excited instability wave in a free shear layer. *J. Fluid Mech.* **197**, 295.

GOLDSTEIN, M. E. & LEE, S. S. 1992 Fully coupled resonant-triad interaction in an adverse-pressure-gradient boundary layer. *J. Fluid Mech.* **245**, 523.

GOLDSTEIN, M. E. & LEIB, S. J. 1988 Nonlinear roll–up of externally excited free shear layers. *J. Fluid Mech.* **191**, 481.

HABERMAN, R. 1972 Critical layers in parallel shear flows. *Stud. Appl. Maths* **50**, 139.

HICKERNELL, F. J. 1984 Time-dependent critical layers in shear flows on the Beta-plane. *J. Fluid Mech.* **142**, 431.

JENNINGS, M. J., STEWART, P. A. & WU, X. 1995 In preparation.

KACHANOV, YU. S. & LEVCHENKO, V. YA. 1984 The resonant interaction of disturbances at laminar–turbulent transition in a boundary layer. *J. Fluid Mech.* **138**, 209.

KHOKHLOV, A. P. 1993 The theory of resonance interaction of Tollmien-Schlichting waves. *Prikl. Mekh. Tekh. Fiz* **4**, 65.

LEE, S. S. 1995 Critical-layer analysis of fully coupled resonant-triad interaction in a boundary layer. Submitted to *J. Fluid Mech.*

MALLIER, R. & MASLOWE, S. A. 1994 Fully coupled resonant triad interactions in a mixing layer. *J. Fluid Mech.* **278**, 101.

MANKBADI, R. R., WU, X. & LEE, S. S. 1993 A critical-layer analysis of the resonant triad in Blasius boundary-layer transition: nonlinear interactions. *J. Fluid Mech.* **256**, 85.

RAETZ, G. S. 1959 A new theory of the cause of transition in fluid flows. Northrop Corp. NOR-59-383 BLC-121.

SARIC, W. S. & THOMAS, A. S. W. 1984 Experiments on subharmonic route to turbulence in boundary layers. In *Turbulence and Chaotic Phenomena in Fluids*, North–Holland.

SMITH, F. T. & STEWART, P. A. 1987 The resonant-triad nonlinear interaction in boundary-layer transition. *J. Fluid Mech.* **179**, 227.

WU, X. 1991 Nonlinear instability of Stokes layers. PhD thesis, University of London.

WU, X. 1992 The nonlinear evolution of high-frequency resonant-triad waves in an oscillatory Stokes-layer at high Reynolds number. *J. Fluid Mech.* **245**, 553.

WU, X. 1993 On critical-layer and diffusion-layer nonlinearity in the three-dimensional stage of boundary-layer transition. *Proc. R. Soc. Lond.* A **433**, 95.

WU, X. & COWLEY, S. J. 1994 On the nonlinear evolution of instability modes in unsteady shear flows: the Stokes layer as a paradigm. *Q. J. Mech. Appl. Maths*, to appear.

WU, X., LEE, S. S. & COWLEY, S. J. 1993 On the weakly nonlinear three-dimensional instability of shear flows to pairs of oblique waves: the Stokes layer as a paradigm. *J. Fluid Mech.* **253**, 681. See also *NASA TM*-105918, *ICOMP*-92-20.

WUNDROW, D. W., HULTGREN, L. S. & GOLDSTEIN, M. E. 1994 Interaction of oblique stability waves with a nonlinear planar wave. *J. Fluid Mech.* **262**, 343.